

ASYMPTOTIC STRUCTURE OF FREE PRODUCT VON NEUMANN ALGEBRAS

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ABSTRACT. Let $(M, \varphi) = (M_1, \varphi_1) * (M_2, \varphi_2)$ be the free product of any σ -finite von Neumann algebras endowed with any faithful normal states. We show that whenever $Q \subset M$ is a von Neumann subalgebra with separable predual such that both Q and $Q \cap M_1$ are the ranges of faithful normal conditional expectations and such that both the intersection $Q \cap M_1$ and the central sequence algebra $Q' \cap M^\omega$ are diffuse (e.g. Q is amenable), then Q must sit inside M_1 . This result generalizes the previous results of the first named author in [Ho14] and moreover completely settles the questions of maximal amenability and maximal property Gamma of the inclusion $M_1 \subset M$ in arbitrary free product von Neumann algebras.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The first class of concrete maximal amenable subalgebras in von Neumann algebras was discovered by Popa in his seminal article [Po83]. He showed that the generator maximal abelian subalgebra $L(\mathbf{Z}) = L(\langle a \rangle)$ is maximal amenable inside the free group factor $L(\mathbf{F}_2) = L(\langle a, b \rangle)$. Popa moreover introduced in [Po83] a powerful method, based on the study of central sequences in the ultraproduct framework, to prove that a given amenable von Neumann subalgebra in a finite von Neumann algebra is maximal amenable. This method will be referred to as Popa's *asymptotic orthogonality property* in this paper. Popa's maximal amenability result [Po83] for free group factors was recently extended by the first named author in [Ho14] to a large class of free product von Neumann algebras, possibly of type III. We refer to [Ho14] and the references therein for further results on maximal amenability in the framework of von Neumann algebras. We point out that Boutonnet-Carderi recently introduced in [BC14] a new method, based on the study of central states, to prove that a given amenable von Neumann subalgebra in a finite von Neumann algebra is maximal amenable. Among other things, they obtained concrete examples of maximal amenable von Neumann subalgebras in type II_1 factors associated with higher rank lattices.

The aim of this paper is to further generalize the recent work of the first named author in [Ho14] and to completely settle the questions of maximal amenability and maximal property Gamma of the inclusion $M_1 \subset M$ arising from an arbitrary free product $(M, \varphi) = (M_1, \varphi_1) * (M_2, \varphi_2)$.

We will say that an inclusion of von Neumann algebras $Q \subset M$ is with *expectation* if there exists a faithful normal conditional expectation from M onto Q . Our first main result is the following optimal *Gamma stability* result inside arbitrary free product von Neumann algebras.

Theorem A. *For each $i \in \{1, 2\}$, let (M_i, φ_i) be any σ -finite von Neumann algebra endowed with a faithful normal state. Denote by $(M, \varphi) = (M_1, \varphi_1) * (M_2, \varphi_2)$ the free product. Let $Q \subset M$ be any von Neumann subalgebra with separable predual and with expectation such that $Q \cap M_1$ is diffuse and with expectation, and furthermore that $Q' \cap M^\omega$ is diffuse. Then we have $Q \subset M_1$.*

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We refer to Theorem 4.3 below for a more general statement that extend [Ho14, Theorem D] to arbitrary free product von Neumann algebras. As a corollary to Theorem A, we infer that amenable von Neumann subalgebras $Q \subset M$ with expectation such that the intersection $Q \cap M_1$ is diffuse and with expectation must in fact sit inside M_1 . Namely, we obtain the following result.

Corollary B. *Let $(M, \varphi) = (M_1, \varphi_1) * (M_2, \varphi_2)$ be as in Theorem A. Let $Q \subset M$ be any amenable von Neumann subalgebra with expectation such that $Q \cap M_1$ is diffuse and with expectation. Then we have $Q \subset M_1$.*

We point out that the separability assumption on the predual of the amenable von Neumann subalgebra $Q \subset M$ is no longer needed in Corollary B. As we mentioned before, in the case when both M_1 and M_2 are *tracial* and both φ_1 and φ_2 are faithful normal *tracial* states, Corollary B is a consequence of [Ho14, Theorem D]. Very recently, Ozawa gave in [Oz15] a short proof of the above Corollary B in the *tracial* case using an idea in [BC14]. However, that proof depends upon the assumption that given states are *tracial*. Moreover, we would like to emphasize that the tools and the techniques we will develop in order to achieve the goal of this paper have strong potential in future research, and indeed lead to our next work on general rigidity phenomenon for free product von Neumann algebras [HU15].

We also point out that [Ho14, Theorem A and Corollary B] hold true under the more general assumption that M_1 is diffuse, instead of the centralizer $(M_1)^{\varphi_1}$ being diffuse as in [Ho14]. In fact, we prove the optimal asymptotic orthogonality property result in arbitrary free product von Neumann algebras (see Theorem 3.1 below) to make those assertions hold under such a general assumption. Remark that this generalization of [Ho14, Theorem A] does not follow from Theorem A, since it is applicable to any intermediate subalgebra $M_1 \subset Q \subset M$ without *a priori* assuming it to be with expectation.

We now briefly explain the strategy of the proof of Theorem A. To simplify the discussion, we will further assume that $Q \subset M$ is a *subfactor*. We refer to Section 4 for further details.

Assume that Q is amenable. In that case, we exploit the fact that Q is AFD with a Cartan subalgebra $A \subset Q$ and hence has lots of central sequences that sit inside the ultraproduct von Neumann subalgebra $A^\omega \subset Q^\omega$. This is a key observation when Q is of type III. Using our generalization of the asymptotic orthogonality property in arbitrary free product von Neumann algebras (see Theorem 3.1 below) and exploiting the recent generalization of Popa's intertwining techniques obtained in [HI15], we then show that any corner of A must embed with expectation into M_1 inside M . By exploiting the regularity property of the Cartan inclusion $A \subset Q$ and using a standard maximality argument, we deduce that $Q \subset M_1$.

We point out that our strategy, based on the study of central sequences in the ultraproduct framework *via* Popa's asymptotic orthogonality property, works for arbitrary von Neumann algebras. Hence we are able to deal with amenable subfactors $Q \subset M$ in Theorem A and Corollary B that can possibly be of type III.

Assume that Q is nonamenable. In that case, we use Connes-Takesaki's structure theory [Co72, Ta03] and Popa's deformation/rigidity theory [Po01, Po03, Po06] inside the ultraproduct of the continuous core $(c_\varphi(M))^\omega$. One of the new features of our proof is to exploit a recent result of Masuda-Tomatsu [MT13] showing that the continuous core of the ultraproduct von Neumann algebra $c_{\varphi^\omega}(M^\omega)$ sits, as an intermediate von Neumann subalgebra with trace preserving conditional expectations, between $c_\varphi(M)$ and $(c_\varphi(M))^\omega$, that is,

$$c_\varphi(M) \subset c_{\varphi^\omega}(M^\omega) \subset (c_\varphi(M))^\omega.$$

Using Popa's spectral gap rigidity principle and intertwining techniques, we then show that any finite corner of $c_\varphi(Q)$ must embed into $c_\varphi(M_1)$ inside the ambient continuous core $c_\varphi(M)$. By a standard maximality argument, we deduce that $c_\varphi(Q) \subset c_\varphi(M_1)$ and hence $Q \subset M_1$.

We point out that we do need to pass to the continuous core $c_\varphi(M)$ in order to make Popa's spectral gap rigidity principle work since we ultimately use Connes's characterization of amenability for *finite* von Neumann algebras [Co75].

We conclude this paper with an appendix in which we give a short proof of an unpublished result due to the second named author showing that Connes's bicentralizer problem has a positive solution for all type III₁ factors arising as free products of von Neumann algebras.

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2. PRELIMINARIES

For any von Neumann algebra M , we will denote by $\mathcal{Z}(M)$ the centre of M , by $\mathcal{U}(M)$ the group of unitaries in M , by $\text{Ball}(M)$ the unit ball of M with respect to the uniform norm $\|\cdot\|_\infty$ and by $(M, L^2(M), J^M, \mathfrak{P}^M)$ the standard form of M . More generally, for any linear subspace $\mathcal{V} \subset M$, we will denote by $\text{Ball}(\mathcal{V})$ the unit ball of \mathcal{V} with respect to the uniform norm $\|\cdot\|_\infty$.

Background on σ -finite von Neumann algebras. Let M be any σ -finite von Neumann algebra with unique predual M_* and $\varphi \in M_*$ any faithful state. We will write $\|x\|_\varphi = \varphi(x^*x)^{1/2}$ for all $x \in M$. Recall that on $\text{Ball}(M)$, the topology given by $\|\cdot\|_\varphi$ coincides with the σ -strong topology. Denote by $\xi_\varphi \in \mathfrak{P}^M$ the unique canonical implementing vector of φ . The mapping $M \rightarrow L^2(M) : x \mapsto x\xi_\varphi$ defines an embedding with dense image such that $\|x\|_\varphi = \|x\xi_\varphi\|_{L^2(M)}$ for all $x \in M$.

We denote by σ^φ the modular automorphism group of the state φ . The centralizer M^φ of the state φ is by definition the fixed point algebra of (M, σ^φ) . The *continuous core* of M with respect to φ , denoted by $c_\varphi(M)$, is the crossed product von Neumann algebra $M \rtimes_{\sigma^\varphi} \mathbf{R}$. The natural inclusion $\pi_\varphi : M \rightarrow c_\varphi(M)$ and the unitary representation $\lambda_\varphi : \mathbf{R} \rightarrow c_\varphi(M)$ satisfy the *covariance* relation

$$\lambda_\varphi(t)\pi_\varphi(x)\lambda_\varphi(t)^* = \pi_\varphi(\sigma_t^\varphi(x)) \quad \text{for all } x \in M \text{ and all } t \in \mathbf{R}.$$

Put $L_\varphi(\mathbf{R}) = \lambda_\varphi(\mathbf{R})''$. There is a unique faithful normal conditional expectation $E_{L_\varphi(\mathbf{R})} : c_\varphi(M) \rightarrow L_\varphi(\mathbf{R})$ satisfying $E_{L_\varphi(\mathbf{R})}(\pi_\varphi(x)\lambda_\varphi(t)) = \varphi(x)\lambda_\varphi(t)$. The faithful normal semifinite weight defined by $f \mapsto \int_{\mathbf{R}} \exp(-s)f(s)ds$ on $L^\infty(\mathbf{R})$ gives rise to a faithful normal semifinite weight Tr_φ on $L_\varphi(\mathbf{R})$ via the Fourier transform. The formula $\text{Tr}_\varphi = \text{Tr}_\varphi \circ E_{L_\varphi(\mathbf{R})}$ extends it to a faithful normal semifinite trace on $c_\varphi(M)$.

Because of Connes's Radon–Nikodym cocycle theorem [Co72, Théorème 1.2.1] (see also [Ta03, Theorem VIII.3.3]), the semifinite von Neumann algebra $c_\varphi(M)$ together with its trace Tr_φ does not depend on the choice of φ in the following precise sense. If ψ is another faithful normal state on M , there is a canonical surjective $*$ -isomorphism $\Pi_{\varphi,\psi} : c_\psi(M) \rightarrow c_\varphi(M)$ such that $\Pi_{\varphi,\psi} \circ \pi_\psi = \pi_\varphi$ and $\text{Tr}_\varphi \circ \Pi_{\varphi,\psi} = \text{Tr}_\psi$. Note however that $\Pi_{\varphi,\psi}$ does not map the subalgebra

$L_\psi(\mathbf{R}) \subset c_\psi(M)$ onto the subalgebra $L_\varphi(\mathbf{R}) \subset c_\varphi(M)$ (and hence we use the symbol $L_\varphi(\mathbf{R})$ instead of the usual $L(\mathbf{R})$).

In order to prove the asymptotic orthogonality property inside arbitrary free product von Neumann algebras (see Theorem 3.1 below), we will use the following key simple lemma whose proof is similar to [MU12, Proposition 2.8] using [HS90, Theorem 11.1].

Lemma 2.1. *Let (M, φ) be any diffuse σ -finite von Neumann algebra endowed with a faithful normal state. For every $\delta > 0$, there exists a faithful normal state $\psi \in M_*$ such that $\|\varphi - \psi\| < \delta$ and such that the centralizer M^ψ is diffuse.*

Proof. Write $M = M_d \oplus M_c$ where M_d is of type I with diffuse center and M_c has no type I direct summand. The above decomposition gives $\varphi = \varphi_d + \varphi_c$. By [HS90, Theorem 11.1] (which dates back to Connes-Størmer's transitivity theorem [CS78]), one can find a faithful normal positive linear functional $\varphi'_c \in (M_c)_*$ such that $\|\varphi'_c\|_{(M_c)_*} = \|\varphi_c\|_{(M_c)_*}$, $\|\varphi_c - \varphi'_c\|_{(M_c)_*} < \delta$ and $(M_c)^{\varphi'_c}$ is of type II_1 . Put $\psi := \varphi_d + \varphi'_c$ and observe that $\psi \in M_*$ is a faithful normal state. Then we have $\|\varphi - \psi\|_{M_*} = \|\varphi_c - \varphi'_c\|_{(M_c)_*} < \delta$ and $\mathcal{Z}(M_d) \oplus (M_c)^{\varphi'_c} \subset M^\psi$. Therefore, the centralizer M^ψ is a diffuse von Neumann subalgebra (see e.g. [Bl06, Theorem IV.2.2.3]). \square

Popa's intertwining techniques. To fix notation, let M be any σ -finite von Neumann algebra, 1_A and 1_B any nonzero projections in M , $A \subset 1_A M 1_A$ and $B \subset 1_B M 1_B$ any von Neumann subalgebras. Popa introduced his powerful *intertwining-by-bimodules techniques* in [Po01] in the case when M is finite and more generally in [Po03] in the case when M is endowed with an almost periodic faithful normal state φ for which $1_A \in M^\varphi$, $A \subset 1_A M^\varphi 1_A$ and $1_B \in M^\varphi$, $B \subset 1_B M^\varphi 1_B$. It was showed in [HV12, Ue12] that Popa's intertwining techniques extend to the case when B is finite and with expectation in $1_B M 1_B$ and $A \subset 1_A M 1_A$ is any von Neumann subalgebra.

In this paper, we will need the following generalization of [Po01, Theorem A.1] in the case when $A \subset 1_A M 1_A$ is any finite von Neumann subalgebra with expectation and $B \subset 1_B M 1_B$ is any von Neumann subalgebra with expectation.

Theorem 2.2 ([HI15, Theorem 4.3]). *Let M be any σ -finite von Neumann algebra, 1_A and 1_B any nonzero projections in M , $A \subset 1_A M 1_A$ and $B \subset 1_B M 1_B$ any von Neumann subalgebras with faithful normal conditional expectations $E_A : 1_A M 1_A \rightarrow A$ and $E_B : 1_B M 1_B \rightarrow B$ respectively. Assume moreover that A is a finite von Neumann algebra.*

Then the following conditions are equivalent:

- (1) *There exist projections $e \in A$ and $f \in B$, a nonzero partial isometry $v \in e M f$ and a unital normal $*$ -homomorphism $\theta : e A e \rightarrow f B f$ such that the inclusion $\theta(e A e) \subset f B f$ is with expectation and $av = v\theta(a)$ for all $a \in e A e$.*
- (2) *There exist $n \geq 1$, a projection $q \in \mathbf{M}_n(B)$, a nonzero partial isometry $v \in (1_A M \otimes \mathbf{M}_{1,n}(\mathbf{C}))q$ and a unital normal $*$ -homomorphism $\pi : A \rightarrow q \mathbf{M}_n(B) q$ such that the inclusion $\pi(A) \subset q \mathbf{M}_n(B) q$ is with expectation and $av = v\pi(a)$ for all $a \in A$.*
- (3) *There exists no net $(w_i)_{i \in I}$ of unitaries in $\mathcal{U}(A)$ such that $E_B(b^* w_i a) \rightarrow 0$ σ -strongly as $i \rightarrow \infty$ for all $a, b \in 1_A M 1_B$.*

If one of the above conditions is satisfied, we will say that A embeds with expectation into B inside M and write $A \preceq_M B$.

Moreover, [HI15, Theorem 4.3] asserts that when $B \subset 1_B M 1_B$ is a *semifinite* von Neumann subalgebra endowed with any fixed faithful normal semifinite trace Tr , then $A \preceq_M B$ if and only if there exist a projection $e \in A$, a Tr -finite projection $f \in B$, a nonzero partial isometry $v \in e M f$ and a unital normal $*$ -homomorphism $\theta : e A e \rightarrow f B f$ such that $av = v\theta(a)$ for all

$a \in eAe$. Hence, in that case, the notation $A \preceq_M B$ is consistent with [Ue12, Proposition 3.1]. In particular, the projection $q \in \mathbf{M}_n(B)$ in Theorem 2.2 (2) is chosen to be finite under the trace $\text{Tr} \otimes \text{tr}_n$, when B is semifinite with any fixed faithful normal semifinite trace Tr . We refer to [HI15, Section 4] for further details.

We say that a σ -finite von Neumann algebra P is *tracial* if it is endowed with a faithful normal tracial state τ . Following [Jo82, PP84], a unital inclusion of tracial von Neumann algebras $A \subset (P, \tau)$ has *finite Jones index* if $\dim_A(L^2(P, \tau)_A) < +\infty$ with the Murray–von Neumann dimension function \dim_A determined by τ . Following [Va07, Appendix A], a unital inclusion of tracial von Neumann algebras $A \subset (P, \tau)$ has *essentially finite index* if there exists a sequence of nonzero projections $(p_n)_n$ in $A' \cap P$ such that the unital inclusion of tracial von Neumann algebras $Ap_n \subset (p_n P p_n, \frac{\tau(p_n \cdot p_n)}{\tau(p_n)})$ has finite Jones index for all $n \in \mathbf{N}$ and $p_n \rightarrow 1$ σ -strongly as $n \rightarrow \infty$.

We will need the following technical lemma about how the intertwining technique behaves with respect to taking subalgebras of essentially finite index.

Lemma 2.3 ([Va07, Lemma 3.9]). *Let M be any σ -finite von Neumann algebra, 1_P and 1_B any nonzero projections in M , $P \subset 1_P M 1_P$ and $B \subset 1_B M 1_B$ any von Neumann subalgebras with expectation. Assume moreover that P is a finite von Neumann algebra and $A \subset P$ is a unital von Neumann subalgebra of essentially finite index. Then $A \preceq_M B$ implies $P \preceq_M B$.*

Proof. This result is [Va07, Lemma 3.9] when the ambient von Neumann algebra M is *finite* and its proof applies *mutatis mutandis* to our more general setting. \square

We will moreover need the following two technical lemmas about intertwining subalgebras inside continuous cores.

Lemma 2.4. *Let (M, φ) be any σ -finite von Neumann algebra endowed with a faithful normal state. Let $q \in M^\varphi$ be any nonzero projection and $Q \subset qMq$ any von Neumann subalgebra that is globally invariant under the modular automorphism group σ^{φ_q} of $\varphi_q = \frac{\varphi(q \cdot q)}{\varphi(q)}$. Denote by E_Q the unique φ_q -preserving conditional expectation from qMq onto Q .*

Then for every nonzero finite trace projection $p \in c_\varphi(M)$ and every net $(u_i)_{i \in I}$ in $\text{Ball}(M)$ such that $\lim_i E_Q(b^ u_i a) = 0$ σ -strongly for all $a, b \in Mq$, we have*

$$\lim_i \|E_{c_{\varphi_q}(Q)}(y^* p \pi_\varphi(u_i) p x)\|_2 = 0, \forall x, y \in p c_\varphi(M) \pi_\varphi(q).$$

In particular, for any faithful normal state $\psi \in M_$, any nonzero projection $r \in M^\psi$, any von Neumann subalgebra $R \subset rM^\psi r$ satisfying $R \not\preceq_M Q$ and any finite trace projection $s \in L_\psi(\mathbf{R})$, we have*

$$\Pi_{\varphi, \psi}(\pi_\psi(R)s) \not\preceq_{c_\varphi(M)} c_{\varphi_q}(Q).$$

Proof. The proof is essentially contained in [BHR12, Proposition 2.10] (see also [HR10, Proposition 5.3]). Simply denote by $\text{Tr} = \text{Tr}_\varphi$ the canonical trace on $c_\varphi(M)$ and by $\|\cdot\|_2$ the L^2 -norm with respect to Tr . Let $x, y \in \text{Ball}(p c_\varphi(M) \pi_\varphi(q))$ be any elements. Fix an increasing sequence $(p_m)_m$ of finite trace projections in $L_\varphi(\mathbf{R})$ such that $p_m \rightarrow 1$ σ -strongly. Observe that $p_m \pi_\varphi(q) = \pi_\varphi(q) p_m$ for all $m \in \mathbf{N}$, since $q \in M^\varphi$.

Let $\varepsilon > 0$. Since $\text{Tr}(p) < +\infty$, we may choose $m \in \mathbf{N}$ large enough such that

$$(2.1) \quad \|px - p x p_m\|_2 + \|y^* p - p_m y^* p\|_2 < \frac{\varepsilon}{2}.$$

Observe that the unital $*$ -subalgebra

$$\mathcal{A} := \left\{ \sum_{j=1}^n \pi_\varphi(a_j) \lambda_\varphi(t_j) : n \geq 1, a_1, \dots, a_n \in M, t_1, \dots, t_n \in \mathbf{R} \right\}$$

is σ -*-strongly dense in $c_\varphi(M)$. Using Kaplansky's density theorem and since $\text{Tr}(p_m) < +\infty$, there exist $x_0, y_0 \in \text{Ball}(\mathcal{A}\pi_\varphi(q))$ such that

$$(2.2) \quad \|p x p_m - x_0 p_m\|_2 + \|p_m y^* p - p_m y_0^*\|_2 < \frac{\varepsilon}{2}.$$

Using (2.1) and (2.2), for all $i \in I$, we have

$$(2.3) \quad \|\mathbb{E}_{c_{\varphi_q}(Q)}(y^* p \pi_\varphi(u_i) p x)\|_2 \leq \|\mathbb{E}_{c_{\varphi_q}(Q)}(p_m y_0^* \pi_\varphi(u_i) x_0 p_m)\|_2 + \varepsilon.$$

Write $x_0 = \sum_{j=1}^\ell \pi_\varphi(a_j) \lambda_\varphi(t_j)$ and $y_0 = \sum_{k=1}^n \pi_\varphi(b_k) \lambda_\varphi(t'_k)$ for some $a_j, b_k \in Mq$ and $t_j, t'_k \in \mathbf{R}$. Since

$$\mathbb{E}_{c_{\varphi_q}(Q)}(p_m y_0^* \pi_\varphi(u_i) x_0 p_m) = \sum_{j,k} p_m \lambda_\varphi(t'_k)^* \pi_\varphi(\mathbb{E}_Q(b_k^* u_i a_j)) \lambda_\varphi(t_j) p_m$$

and since $\lim_i \mathbb{E}_Q(b_k^* u_i a_j) = 0$ σ -strongly for all j, k and since $\text{Tr}(p_m) < +\infty$, we obtain

$$(2.4) \quad \lim_i \|\mathbb{E}_{c_{\varphi_q}(Q)}(p_m y_0^* \pi_\varphi(u_i) x_0 p_m)\|_2 = 0.$$

Then (2.3) and (2.4) imply that $\limsup_i \|\mathbb{E}_{c_{\varphi_q}(Q)}(y^* p \pi_\varphi(u_i) p x)\|_2 \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we finally obtain

$$\lim_i \|\mathbb{E}_{c_{\varphi_q}(Q)}(y^* p \pi_\varphi(u_i) p x)\|_2 = 0.$$

Next, assume that $\psi \in M_*$ is any faithful normal state, $r \in M^\psi$ is any nonzero projection, $R \subset rM^\psi r$ is any von Neumann subalgebra such that $R \not\prec_M Q$ and $s \in L_\psi(\mathbf{R})$ is any nonzero finite trace projection. By Theorem 2.2, there exists a net $(v_i)_{i \in J}$ in $\mathcal{U}(R)$ such that $\lim_j \|\mathbb{E}_Q(b^* v_j a)\|_\varphi = 0$ for all $a, b \in rMq$. Recall that $\Pi_{\varphi, \psi} \circ \pi_\psi = \pi_\varphi$ and $\text{Tr}_\varphi \circ \Pi_{\varphi, \psi} = \text{Tr}_\psi$. Put $p = \Pi_{\varphi, \psi}(s)$. The first part of the proof implies that $\lim_j \|\mathbb{E}_{c_{\varphi_q}(Q)}(y^* \pi_\varphi(v_j) p x)\|_2 = 0$ for all $x, y \in p\pi_\varphi(r)c_\varphi(M)\pi_\varphi(q)$. Since $\pi_\varphi(v_j)p = \Pi_{\varphi, \psi}(\pi_\psi(v_j)s) \in \mathcal{U}(\Pi_{\varphi, \psi}(\pi_\psi(R)s))$ for all $j \in J$, we obtain that $\Pi_{\varphi, \psi}(\pi_\psi(R)s) \not\prec_{c_\varphi(M)} c_{\varphi_q}(Q)$ by Theorem 2.2. \square

Lemma 2.5. *Let M be any σ -finite von Neumann algebra and $\varphi, \psi \in M_*$ any faithful states. Let $q \in M^\psi$ be any nonzero projection and $Q \subset qMq$ any diffuse von Neumann subalgebra that is globally invariant under the modular automorphism group σ^{ψ_q} of $\psi_q = \frac{\psi(q \cdot q)}{\psi(q)}$. Then for every nonzero finite trace projection $p \in L_\psi(\mathbf{R})$, we have $c_{\psi_q}(Q) \subset c_\psi(M)$ naturally and*

$$\Pi_{\varphi, \psi}(p c_{\psi_q}(Q) p) \not\prec_{c_\varphi(M)} L_\varphi(\mathbf{R}).$$

Proof. Denote by $z \in \mathcal{Z}(Q)$ the unique central projection such that Qz is of type I and Qz^\perp has no type I direct summand. Observe that $z \in M^\psi$, $\pi_\psi(z) \in \mathcal{Z}(c_{\psi_q}(Q))$ and

$$p c_{\psi_q}(Q) p = p c_{\psi_q}(Q) p \pi_\psi(z) \oplus p c_{\psi_q}(Q) p \pi_\psi(z^\perp) = p c_{\psi_q}(Q) p \pi_\psi(z) \oplus p c_{\psi_q}(Q) p \pi_\psi(z^\perp) p.$$

Since Qz^\perp has no type I direct summand, $p c_{\psi_q}(Q) p \pi_\psi(z^\perp) p$ has no type I direct summand either. This follows from the fact that continuous cores are independent of states or even weights (due to Connes's Radon–Nykodym cocycle theorem [Co72, Théorème 1.2.1]) as well as the fact that the continuous core of any type III von Neumann algebra must be of type II_∞ (see [Ta03, Theorem XII.1.1]). Hence we have

$$\Pi_{\varphi, \psi}(p c_{\psi_q}(Q) p \pi_\psi(z^\perp)) \not\prec_{c_\varphi(M)} L_\varphi(\mathbf{R}).$$

Since Qz is of type I and diffuse, $\mathcal{Z}(Qz) \subset Q^{\psi_q} z = (Qz)^{\psi_z}$ with $\psi_z := \frac{\psi(z \cdot z)}{\psi(z)}$ is also diffuse and hence $\mathcal{Z}(Qz) \not\prec_M \mathbf{C}1$. Then Lemma 2.4 (with letting the Q there be the trivial algebra) implies that

$$\Pi_{\varphi, \psi}(p c_{\psi_q}(Q) p \pi_\psi(z)) \not\prec_{c_\varphi(M)} L_\varphi(\mathbf{R}).$$

Combining the above two facts, we finally obtain that $\Pi_{\varphi, \psi}(p c_{\psi_q}(Q) p) \not\prec_{c_\varphi(M)} L_\varphi(\mathbf{R})$. \square

Amalgamated free product von Neumann algebras. For each $i \in \{1, 2\}$, let $B \subset M_i$ be any inclusion of σ -finite von Neumann algebras with faithful normal conditional expectation $E_i : M_i \rightarrow B$. The *amalgamated free product* $(M, E) = (M_1, E_1) *_B (M_2, E_2)$ is a pair of von Neumann algebra M generated by M_1 and M_2 and faithful normal conditional expectation $E : M \rightarrow B$ such that M_1, M_2 are *freely independent* with respect to E :

$$E(x_1 \cdots x_n) = 0 \text{ whenever } x_j \in M_{i_j}^\circ \text{ and } i_1 \neq \cdots \neq i_n.$$

Here and in what follows, we denote by $M_i^\circ = \ker(E_i)$. We refer to the product $x_1 \cdots x_n$ where $x_j \in M_{i_j}^\circ$ and $i_1 \neq \cdots \neq i_n$ as a *reduced word* in $M_{i_1}^\circ \cdots M_{i_n}^\circ$ of *length* $n \geq 1$. The linear span of B and of all the reduced words in $M_{i_1}^\circ \cdots M_{i_n}^\circ$ where $n \geq 1$ and $i_1 \neq \cdots \neq i_n$ forms a unital σ -strongly dense $*$ -subalgebra of M . We call the resulting M the *amalgamated free product von Neumann algebra* of (M_1, E_1) and (M_2, E_2) .

When $B = \mathbf{C}1$, $E_i = \varphi_i(\cdot)1$ for all $i \in \{1, 2\}$ and $E = \varphi(\cdot)1$, we will simply denote by $(M, \varphi) = (M_1, \varphi_1) * (M_2, \varphi_2)$ and call the resulting M the *free product von Neumann algebra* of (M_1, φ_1) and (M_2, φ_2) .

When B is a semifinite von Neumann algebra with faithful normal semifinite trace Tr and the weight $\text{Tr} \circ E_i$ is tracial on M_i for every $i \in \{1, 2\}$, then the weight $\text{Tr} \circ E$ is tracial on M (see [Po90, Proposition 3.1] for the finite case and [Ue98a, Theorem 2.6] for the general case). In particular, M is a semifinite von Neumann algebra. In that case, we will refer to $(M, E) = (M_1, E_1) *_B (M_2, E_2)$ as a *semifinite amalgamated free product*.

Let $\varphi \in B_*$ be any faithful normal state. Then for all $t \in \mathbf{R}$, we have $\sigma_t^{\varphi \circ E} = \sigma_t^{\varphi \circ E_1} * \sigma_t^{\varphi \circ E_2}$ (see [Ue98a, Theorem 2.6]). By [Ta03, Theorem IX.4.2], there exists a unique $\varphi \circ E$ -preserving conditional expectation $E_{M_1} : M \rightarrow M_1$. Moreover, we have $E_{M_1}(x_1 \cdots x_n) = 0$ for all the reduced words $x_1 \cdots x_n$ that contain at least one letter from M_2° (see e.g. [Ue10, Lemma 2.1]). We will denote by $M \ominus M_1 = \ker(E_{M_1})$. For more on (amalgamated) free product von Neumann algebras, we refer the reader to [BHR12, Po90, Ue98a, Ue10, Ue12, Vo85, VDN92].

Lemma 2.6. *For each $i \in \{1, 2\}$, let $B \subset M_i$ be any inclusion of σ -finite von Neumann algebras with faithful normal conditional expectations $E_i : M_i \rightarrow B$. Denote by $(M, E) = (M_1, E_1) *_B (M_2, E_2)$ the amalgamated free product.*

Let $\psi \in M_$ be any faithful normal state such that $\psi = \psi \circ E_{M_1}$. Let $(u_j)_{j \in J}$ be any net in $\text{Ball}((M_1)^\psi)$ such that $\lim_j E_1(b^* u_j a) = 0$ σ -strongly for all $a, b \in M_1$. Then for all $x, y \in M \ominus M_1$, we have that $\lim_j E_{M_1}(y^* u_j x) = 0$ σ -strongly.*

Proof. We first prove the σ -strong convergence when $x, y \in M_1 M_2^\circ \cdots M_2^\circ M_1$ are *words* of the form $x = ax'c$ and $y = by'd$ with $a, b, c, d \in M_1$ and $x', y' \in M_2^\circ \cdots M_2^\circ$. By freeness with amalgamation over B , for all $n \in \mathbf{N}$, we have

$$E_{M_1}(y^* u_j x) = E_{M_1}(d^* y'^* b^* u_j a x' c) = E_{M_1}(d^* y'^* E_1(b^* u_j a) x' c).$$

Since $\lim_j E_1(b^* u_j a) = 0$ σ -strongly, we have $\lim_j E_{M_1}(y^* u_j x) = 0$ σ -strongly.

Recall that $\psi = \psi \circ E_{M_1}$. We next prove the σ -strong convergence when $x \in M \ominus M_1$ is any element and $y \in M_1 M_2^\circ \cdots M_2^\circ M_1$ is any *word* as above. Indeed, we may choose a sequence $(x_k)_k$, where each x_k is a finite linear combination of *words* in $M_1 M_2^\circ \cdots M_2^\circ M_1$, and such that $\lim_{k \rightarrow \infty} \|x - x_k\|_\psi = 0$. Then by triangle inequality, for all $j \in J$ and $k \in \mathbf{N}$, we have

$$\begin{aligned} \|E_{M_1}(y^* u_j x)\|_\psi &\leq \|E_{M_1}(y^* u_j x_k)\|_\psi + \|E_{M_1}(y^* u_j (x - x_k))\|_\psi \\ &\leq \|E_{M_1}(y^* u_j x_k)\|_\psi + \|y^* u_j (x - x_k)\|_\psi \\ &\leq \|E_{M_1}(y^* u_j x_k)\|_\psi + \|y\|_\infty \|x - x_k\|_\psi. \end{aligned}$$

The first part of the proof implies that $\limsup_j \|E_{M_1}(y^* u_j x)\|_\psi \leq \|y\|_\infty \|x - x_k\|_\psi$ for all $k \in \mathbf{N}$ and hence $\lim_j \|E_{M_1}(y^* u_j x)\|_\psi = 0$.

Recall that $\psi = \psi \circ E_{M_1}$ and hence $\sigma_t^\psi(M_1) = M_1$ for all $t \in \mathbf{R}$. We next prove the σ -strong convergence when $x \in M \ominus M_1$ is any analytic element with respect to the modular automorphism group σ^ψ and $y \in M \ominus M_1$ is any element. Indeed, we may choose a sequence $(y_k)_k$, where each y_k is a finite linear combination of *words* in $M_1 M_2^\circ \cdots M_2^\circ M_1$, and such that $\lim_{k \rightarrow \infty} \|y^* - y_k^*\|_\psi = 0$. Then by triangle inequality, for all $j \in J$ and all $k \in \mathbf{N}$, we have

$$\begin{aligned} \|E_{M_1}(y^* u_j x)\|_\psi &\leq \|E_{M_1}(y_k^* u_j x)\|_\psi + \|E_{M_1}((y^* - y_k^*) u_j x)\|_\psi \\ &\leq \|E_{M_1}(y_k^* u_j x)\|_\psi + \|(y^* - y_k^*) u_j x\|_\psi \\ &= \|E_{M_1}(y_k^* u_j x)\|_\psi + \|J^M \sigma_{i/2}^\psi(x)^* u_j^* J^M (y^* - y_k^*)\|_\psi \\ &= \|E_{M_1}(y_k^* u_j x)\|_\psi + \|\sigma_{i/2}^\psi(x)\|_\infty \|y^* - y_k^*\|_\psi. \end{aligned}$$

The second part of the proof implies that $\limsup_j \|E_{M_1}(y^* u_j x)\|_\psi \leq \|\sigma_{i/2}^\psi(x)\|_\infty \|y^* - y_k^*\|_\psi$ for all $k \in \mathbf{N}$ and hence $\lim_j \|E_{M_1}(y^* u_j x)\|_\psi = 0$.

We finally prove the σ -strong convergence when $x, y \in M \ominus M_1$ are any elements. Indeed, we may choose a sequence $(x_k)_k$ in $M \ominus M_1$ of analytic elements with respect to the modular automorphism group σ^ψ such that $\lim_{k \rightarrow \infty} \|x - x_k\|_\psi = 0$. Then by triangle inequality, for all $j \in J$ and all $k \in \mathbf{N}$, we have

$$\begin{aligned} \|E_{M_1}(y^* u_j x)\|_\psi &\leq \|E_{M_1}(y^* u_j x_k)\|_\psi + \|E_{M_1}(y^* u_j (x - x_k))\|_\psi \\ &\leq \|E_{M_1}(y^* u_j x_k)\|_\psi + \|y^* u_j (x - x_k)\|_\psi \\ &\leq \|E_{M_1}(y^* u_j x_k)\|_\psi + \|y\|_\infty \|x - x_k\|_\psi. \end{aligned}$$

The third part of the proof implies that $\limsup_j \|E_{M_1}(y^* u_j x)\|_\psi \leq \|y\|_\infty \|x - x_k\|_\psi$ for all $k \in \mathbf{N}$ and hence $\lim_j \|E_{M_1}(y^* u_j x)\|_\psi = 0$. This finishes the proof of Lemma 2.6. \square

The next proposition about controlling the (*quasi*)-normalizer of diffuse subalgebras inside free product von Neumann algebras will be very useful in the proof of Theorem A. This is a variant of [IPP05, Theorem 1.1] and [Ue12, Proposition 3.3], but the proof uses an idea of [Va06, Lemma D3] and the previous lemma crucially. We point out that the first assertion also generalizes [Ue10, Corollary 3.2] (with $n = 1$, $\pi(x) = uxu^*$ and $v = u$ for $u \in \mathcal{U}(A' \cap M)$ or $\mathcal{N}_M(A)$). A more general, unified statement seems possible in the framework of amalgamated free products because the previous lemma is quite general, but the statements below fit the later use.

Proposition 2.7. *For each $i \in \{1, 2\}$, let (M_i, φ_i) be any σ -finite von Neumann algebra endowed with a faithful normal state. Denote by $(M, \varphi) = (M_1, \varphi_1) * (M_2, \varphi_2)$ the free product.*

- (1) *Let $1_Q \in M_1$ be any nonzero projection and $Q \subset 1_Q M_1 1_Q$ any diffuse von Neumann subalgebra with expectation. For every $n \geq 1$, every (not necessarily unital) normal $*$ -homomorphism $\pi : Q \rightarrow \mathbf{M}_n(M_1)$ and every nonzero partial isometry $v \in (1_Q M \otimes \mathbf{M}_{1,n}(\mathbf{C}))\pi(1_Q)$ such that $xv = v\pi(x)$ for all $x \in Q$, we have*

$$v \in (1_Q M_1 \otimes \mathbf{M}_{1,n}(\mathbf{C}))\pi(1_Q).$$

- (2) *Let $1_A \in M$ be any nonzero projection and $A \subset 1_A M 1_A$ any diffuse von Neumann subalgebra with expectation. For every $n \geq 1$, every (not necessarily unital) normal $*$ -homomorphism $\pi : A \rightarrow \mathbf{M}_n(M_1)$ such that the inclusion $\pi(A) \subset \pi(1_A)\mathbf{M}_n(M_1)\pi(1_A)$ is with expectation and every nonzero partial isometry $v \in (1_A M \otimes \mathbf{M}_{1,n}(\mathbf{C}))\pi(1_A)$ such that $av = v\pi(a)$ for all $a \in A$, we have*

$$v^* v \in \pi(1_A)\mathbf{M}_n(M_1)\pi(1_A) \quad \text{and} \quad v^* \mathcal{N}_{1_A M 1_A}(A)'' v \subset v^* v \mathbf{M}_n(M_1) v^* v.$$

Proof. (1) As in the proof of Lemma 2.1 and since $Q \subset 1_Q M_1 1_Q$ is with expectation, we may choose a faithful normal state $\psi \in M_*$ such that $\psi = \psi \circ E_{M_1}$, $1_Q \in (M_1)^\psi$, $Q \subset 1_Q M 1_Q$

is globally invariant under the modular automorphism group σ^{ψ_Q} and $Q^{\psi_Q} \subset 1_Q(M_1)^{\psi} 1_Q$ is diffuse where $\psi_Q := \frac{\psi(1_Q \cdot 1_Q)}{\psi(1_Q)}$.

Let n, π, v as in the statement. Denote by tr_n the canonical normalized trace on $\mathbf{M}_n(\mathbf{C})$ and write $v = [v_1 \cdots v_n] \in (1_Q M \otimes \mathbf{M}_{1,n}(\mathbf{C}))\pi(1_Q)$. For all $x \in Q$, since $xv = v\pi(x)$, we have

$$x E_{\mathbf{M}_n(M_1)}(v) = E_{\mathbf{M}_n(M_1)}(xv) = E_{\mathbf{M}_n(M_1)}(v\pi(x)) = E_{\mathbf{M}_n(M_1)}(v)\pi(x)$$

and hence

$$(2.5) \quad x(v - E_{\mathbf{M}_n(M_1)}(v)) = (v - E_{\mathbf{M}_n(M_1)}(v))\pi(x).$$

Put $w := v - E_{\mathbf{M}_n(M_1)}(v) \in (1_Q(M \ominus M_1) \otimes \mathbf{M}_{1,n}(\mathbf{C}))\pi(1_Q)$ and write $w = [w_1 \cdots w_n]$ with $w_1, \dots, w_n \in 1_Q(M \ominus M_1)$. Fix a sequence of unitaries $(u_k)_k$ in $\mathcal{U}(Q^{\psi_Q})$ such that $\lim_{k \rightarrow \infty} u_k = 0$ σ -weakly. By Lemma 2.6, we have

$$(2.6) \quad \lim_{k \rightarrow \infty} \|E_{\mathbf{M}_n(M_1)}(w^* u_k w)\|_{\psi \otimes \text{tr}_n}^2 = \lim_{k \rightarrow \infty} \sum_{i,j=1}^n \|E_{M_1}(w_i^* u_k w_j)\|_{\psi}^2 = 0.$$

Using (2.5) and (2.6) and since $\pi(u_k) \in \mathcal{U}(\pi(Q))$ and $w^* w \in \pi(Q)' \cap \pi(1_Q)\mathbf{M}_n(M)\pi(1_Q)$, we have

$$\begin{aligned} \|E_{\mathbf{M}_n(M_1)}(w^* w)\|_{\psi \otimes \text{tr}_n} &= \limsup_{k \rightarrow \infty} \|\pi(u_k) E_{\mathbf{M}_n(M_1)}(w^* w)\|_{\psi \otimes \text{tr}_n} \\ &= \limsup_{k \rightarrow \infty} \|E_{\mathbf{M}_n(M_1)}(\pi(u_k) w^* w)\|_{\psi \otimes \text{tr}_n} \\ &= \limsup_{k \rightarrow \infty} \|E_{\mathbf{M}_n(M_1)}(w^* w \pi(u_k))\|_{\psi \otimes \text{tr}_n} \\ &= \lim_{k \rightarrow \infty} \|E_{\mathbf{M}_n(M_1)}(w^* u_k w)\|_{\psi \otimes \text{tr}_n} \\ &= 0. \end{aligned}$$

This implies that $w^* w = 0$ and hence $w = 0$. Thus $v = E_{\mathbf{M}_n(M_1)}(v) \in (1_Q M_1 \otimes \mathbf{M}_{1,n}(\mathbf{C}))\pi(1_Q)$.

(2) We will be working inside the amalgamated free product von Neumann algebra

$$\mathbf{M}_n(M) = (\mathbf{M}_n(M_1), \varphi_1 \otimes \text{id}_n) *_{\mathbf{M}_n(\mathbf{C})} (\mathbf{M}_n(M_2), \varphi_2 \otimes \text{id}_n),$$

and substitute formula (2.7) below for the assumption of item (1) that $xv = v\pi(x)$ for all $x \in Q$.

Since $\pi(A) \subset \pi(1_A)\mathbf{M}_n(M_1)\pi(1_A)$ is a diffuse von Neumann subalgebra with expectation, we may choose, as in the proof of item (1), a faithful normal state $\psi \in \mathbf{M}_n(M)_*$ such that $\psi = \psi \circ E_{\mathbf{M}_n(M_1)}$, $\pi(1_A) \in \mathbf{M}_n(M_1)^{\psi}$ and $\pi(A) \cap \pi(1_A)\mathbf{M}_n(M_1)^{\psi}\pi(1_A)$ is diffuse. Fix a sequence of unitaries $(u_k)_k$ in $\pi(A) \cap \pi(1_A)\mathbf{M}_n(M_1)^{\psi}\pi(1_A)$ such that $\lim_{k \rightarrow \infty} u_k = 0$ σ -weakly. For each $k \in \mathbf{N}$, we may write $u_k = \pi(a_k)$ with a unitary $a_k \in A$.

Let now $x \in \mathcal{N}_{1_A M_1 A}(A)$ be any normalizing unitary element. Then for all $a \in A$, we have

$$(2.7) \quad v^* x v \pi(a) = v^* x a v = v^* (x a x^*) x v = \pi(x a x^*) v^* x v,$$

and hence, as in the proof of item (1), for every $k \in \mathbf{N}$ we have

$$(2.8) \quad (v^* x v - E_{\mathbf{M}_n(M_1)}(v^* x v)) u_k = \pi(x a_k x^*) (v^* x v - E_{\mathbf{M}_n(M_1)}(v^* x v)).$$

Put $w := v^* x v - E_{\mathbf{M}_n(M_1)}(v^* x v) \in \pi(1_A)(\mathbf{M}_n(M) \ominus \mathbf{M}_n(M_1))\pi(1_A)$. Using (2.8) and Lemma 2.6 and since $\pi(x a_k x^*) \in \mathcal{U}(\pi(A))$, we obtain, as in the proof of item (1), that

$$\|E_{\mathbf{M}_n(M_1)}(w w^*)\|_{\psi} = \lim_{k \rightarrow \infty} \|E_{\mathbf{M}_n(M_1)}(w u_k w^*)\|_{\psi} = 0,$$

implying that $v^* x v = E_{\mathbf{M}_n(M_1)}(v^* x v) \in \mathbf{M}_n(M_1)$ and the desired assertion is immediate. \square

Ultraproduct von Neumann algebras. Let M be any σ -finite von Neumann algebra. Define

$$\begin{aligned}\mathcal{I}_\omega(M) &= \{(x_n)_n \in \ell^\infty(\mathbf{N}, M) : x_n \rightarrow 0 \text{ }^*\text{-strongly as } n \rightarrow \omega\}, \\ \mathcal{M}^\omega(M) &= \{(x_n)_n \in \ell^\infty(\mathbf{N}, M) : (x_n)_n \mathcal{I}_\omega(M) \subset \mathcal{I}_\omega(M) \text{ and } \mathcal{I}_\omega(M)(x_n)_n \subset \mathcal{I}_\omega(M)\}.\end{aligned}$$

We have that the *multiplier algebra* $\mathcal{M}^\omega(M)$ is a C^* -algebra and $\mathcal{I}_\omega(M) \subset \mathcal{M}^\omega(M)$ is a norm closed two-sided ideal. Following [Oc85], we define the *ultraproduct von Neumann algebra* M^ω by $M^\omega = \mathcal{M}^\omega(M)/\mathcal{I}_\omega(M)$. We denote the image of $(x_n)_n \in \mathcal{M}^\omega(M)$ by $(x_n)^\omega \in M^\omega$.

For all $x \in M$, the constant sequence $(x)_n$ lies in the multiplier algebra $\mathcal{M}^\omega(M)$. We will then identify M with $(M + \mathcal{I}_\omega(M))/\mathcal{I}_\omega(M)$ and regard $M \subset M^\omega$ as a von Neumann subalgebra. The map $E_\omega : M^\omega \rightarrow M : (x_n)^\omega \mapsto \sigma\text{-weak } \lim_{n \rightarrow \omega} x_n$ is a faithful normal conditional expectation. For every faithful normal state $\varphi \in M_*$, the formula $\varphi^\omega = \varphi \circ E_\omega$ defines a faithful normal state on M^ω . Observe that $\varphi^\omega((x_n)^\omega) = \lim_{n \rightarrow \omega} \varphi(x_n)$ for all $(x_n)^\omega \in M^\omega$.

Let $Q \subset M$ be any von Neumann subalgebra with faithful normal conditional expectation $E_Q : M \rightarrow Q$. Choose a faithful normal state $\varphi \in M_*$ such that $\varphi = \varphi \circ E_Q$. We have $\ell^\infty(\mathbf{N}, Q) \subset \ell^\infty(\mathbf{N}, M)$, $\mathcal{I}_\omega(Q) \subset \mathcal{I}_\omega(M)$ and $\mathcal{M}^\omega(Q) \subset \mathcal{M}^\omega(M)$. We will then identify $Q^\omega = \mathcal{M}^\omega(Q)/\mathcal{I}_\omega(Q)$ with $(\mathcal{M}^\omega(Q) + \mathcal{I}_\omega(M))/\mathcal{I}_\omega(M)$ and regard $Q^\omega \subset M^\omega$ as a von Neumann subalgebra. Observe that the norm $\|\cdot\|_{(\varphi|_Q)^\omega}$ on Q^ω is the restriction of the norm $\|\cdot\|_{\varphi^\omega}$ to Q^ω . Observe moreover that $(E_Q(x_n))_n \in \mathcal{I}_\omega(Q)$ for all $(x_n)_n \in \mathcal{I}_\omega(M)$ and $(E_Q(x_n))_n \in \mathcal{M}^\omega(Q)$ for all $(x_n)_n \in \mathcal{M}^\omega(M)$. Therefore, the mapping $E_{Q^\omega} : M^\omega \rightarrow Q^\omega : (x_n)^\omega \mapsto (E_Q(x_n))^\omega$ is a well-defined conditional expectation satisfying $\varphi^\omega \circ E_{Q^\omega} = \varphi^\omega$. Hence, $E_{Q^\omega} : M^\omega \rightarrow Q^\omega$ is a faithful normal conditional expectation.

Put $\mathcal{H} = L^2(M)$. The *ultraproduct Hilbert space* \mathcal{H}^ω is defined to be the quotient of $\ell^\infty(\mathbf{N}, \mathcal{H})$ by the subspace consisting in sequences $(\xi_n)_n$ satisfying $\lim_{n \rightarrow \omega} \|\xi_n\|_{\mathcal{H}} = 0$. We denote the image of $(\xi_n)_n \in \ell^\infty(\mathbf{N}, \mathcal{H})$ by $(\xi_n)^\omega \in \mathcal{H}^\omega$. The inner product space structure on the Hilbert space \mathcal{H}^ω is defined by $\langle (\xi_n)^\omega, (\eta_n)^\omega \rangle_{\mathcal{H}^\omega} = \lim_{n \rightarrow \omega} \langle \xi_n, \eta_n \rangle_{\mathcal{H}}$. The standard Hilbert space $L^2(M^\omega)$ can be embedded into \mathcal{H}^ω as a closed subspace *via* the mapping $L^2(M^\omega) \rightarrow \mathcal{H}^\omega : (x_n)^\omega \xi_{\varphi^\omega} \mapsto (x_n \xi_\varphi)^\omega$. For more on ultraproduct von Neumann algebras, we refer the reader to [AH12, Oc85].

In Section 4, we will need the following well-known fact about ultraproducts of semifinite von Neumann algebras. Let (M, Tr) be any semifinite σ -finite von Neumann endowed with a faithful normal semifinite trace. Then the ultraproduct von Neumann algebra M^ω is semifinite and the weight $\text{Tr} \circ E_\omega$ is tracial on M^ω (see [AH12, Lemma 4.26]).

In Appendix A, we will need the following result about the centralizer $(M^\omega)^{\varphi^\omega}$ of the ultraproduct state φ^ω .

Proposition 2.8. *Let (M, φ) be any σ -finite von Neumann algebra endowed with a faithful normal state and $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$ any nonprincipal ultrafilter.*

- (1) *If $M \neq \mathbf{C}1$, then $(M^\omega)^{\varphi^\omega} \neq \mathbf{C}1$.*
- (2) *If M is diffuse, then $(M^\omega)^{\varphi^\omega}$ is diffuse.*

Proof. (1) Assume that $M \neq \mathbf{C}1$. If $M^\varphi \neq \mathbf{C}1$, then we also have $(M^\omega)^{\varphi^\omega} \neq \mathbf{C}1$ since $M^\varphi \subset (M^\omega)^{\varphi^\omega}$. If $M^\varphi = \mathbf{C}1$, then M is a type III_1 factor by [AH12, Lemma 5.3]. By [AH12, Theorem 4.20], $(M^\omega)^{\varphi^\omega}$ is a type II_1 factor and hence $(M^\omega)^{\varphi^\omega} \neq \mathbf{C}1$.

(2) Fix a sequence $(z_n)_n$ of central projections in $\mathcal{Z}(M)$ such that $\sum_n z_n = 1$, Mz_0 has a diffuse center and Mz_n is a diffuse factor for every $n \geq 1$. Observe that $\mathcal{Z}(Mz_0) \subset M^\varphi z_0$ and hence $M^\varphi z_0$ is diffuse. Next, fix $n \geq 1$ such that $z_n \neq 0$ and put $\varphi_{z_n} = \frac{\varphi(z_n \cdot z_n)}{\varphi(z_n)} \in (Mz_n)_*$. If Mz_n is a semifinite factor, then $M^\varphi z_n = (Mz_n)^{\varphi_{z_n}}$ is diffuse. If Mz_n is a type III_λ factor, with $0 \leq \lambda < 1$, then $M^\varphi z_n = (Mz_n)^{\varphi_{z_n}}$ is diffuse by [Co72, Théorème 4.2.1 and Théorème 5.2.1]. If Mz_n is a type III_1 factor, then $(M^\omega)^{\varphi^\omega} z_n = ((Mz_n)^\omega)^{\varphi_{z_n}^\omega}$ is a type II_1 factor by [AH12,

Theorem 4.20]. We finally obtain that $(M^\omega)^{\varphi^\omega} z_n = ((M z_n)^\omega)^{\varphi_{z_n}^\omega}$ is diffuse for all n and hence $(M^\omega)^{\varphi^\omega}$ is diffuse. \square

3. ASYMPTOTIC ORTHOGONALITY PROPERTY

The phenomenon of *asymptotic orthogonality property* inside free group factors was discovered by Popa in his seminal work [Po83, Lemma 2.1]. The main result of this section is the following optimal asymptotic orthogonality property result inside arbitrary free product von Neumann algebras. To fix notation, for each $i \in \{1, 2\}$, let (M_i, φ_i) be any σ -finite von Neumann algebra endowed with a faithful normal state. Denote by $(M, \varphi) = (M_1, \varphi_1) * (M_2, \varphi_2)$ the free product. As usual, denote by $E_{M_1} : M \rightarrow M_1$ the unique φ -preserving conditional expectation. Let $Q \subset M_1$ be any diffuse von Neumann subalgebra with expectation. Fix a faithful state $\psi \in M_*$ such that $\sigma_t^\psi(Q) = Q$ and $\sigma_t^\psi(M_1) = M_1$ for all $t \in \mathbf{R}$. Observe that $\psi = \psi \circ E_{M_1}$.

Theorem 3.1 below is a simultaneous generalization of [Ue10, Proposition 3.5] (which only deals with $y \in \ker(\varphi_2)$) and [Ho14, Theorem 3.1] (which requires the centralizer $(M_1)^{\varphi_1}$ to be diffuse).

Theorem 3.1. *Keep the same notation as above. For all $x \in Q' \cap M^\omega$ and all $y, z \in M \ominus M_1$, the vectors*

$$y(x - E_{M_1^\omega}(x))\xi_{\psi^\omega}, (yE_{M_1^\omega}(x) - E_{M_1^\omega}(x)z)\xi_{\psi^\omega} \text{ and } (E_{M_1^\omega}(x) - x)z\xi_{\psi^\omega}$$

are mutually orthogonal in the standard Hilbert space $L^2(M^\omega)$ where $\xi_{\psi^\omega} \in \mathfrak{P}^{M^\omega}$ is the canonical representing vector of the ultraproduct state ψ^ω .

Proof. The proof of Theorem 3.1 is a reconstruction of [Ho14, Theorem 3.1] and the new input is the ‘state replacement’ procedure developed in [Ue10].

Let $(M^\omega, L^2(M^\omega), J^{M^\omega}, \mathfrak{P}^{M^\omega})$ be the standard form of the ultraproduct von Neumann algebra M^ω , which is known to be obtained from the standard form $(M, L^2(M), J^M, \mathfrak{P}^M)$ of the original von Neumann algebra M in a rather canonical fashion (see [AH12, Corollary 3.27]). It suffices to prove, instead of the original assertion, that, for all $z' \in M \ominus M_1$ with the given x, y in the original assertion, the vectors

$$y(x - E_{M_1^\omega}(x))\xi_{\psi^\omega}, (yE_{M_1^\omega}(x) - J^{M^\omega} z' J^{M^\omega} E_{M_1^\omega}(x))\xi_{\psi^\omega} \text{ and } J^{M^\omega} z' J^{M^\omega} (E_{M_1^\omega}(x) - x)\xi_{\psi^\omega}$$

are mutually orthogonal in the standard Hilbert space $L^2(M^\omega)$. In fact, by a standard approximation argument we may and do assume that the given z in the original assertion is analytic with respect to the modular automorphism group σ^ψ . By [AH12, Theorem 4.1] together with [Ta03, Lemma VIII.3.18 (ii)], we have

$$\begin{aligned} E_{M_1^\omega}(x)z\xi_{\psi^\omega} &= J^{M^\omega} \sigma_{i/2}^\psi(z)^* J^{M^\omega} E_{M_1^\omega}(x)\xi_{\psi^\omega}, \\ (E_{M_1^\omega}(x) - x)z\xi_{\psi^\omega} &= J^{M^\omega} \sigma_{i/2}^\psi(z)^* J^{M^\omega} (E_{M_1^\omega}(x) - x)\xi_{\psi^\omega}, \end{aligned}$$

so that the above new assertion immediately gives the desired one.

For all $i \in \{1, 2\}$, denote by $A_i \subset M_i$ the σ -weakly dense unital $*$ -subalgebra of all the analytic elements in M_i with respect to the modular automorphism group σ^{φ_i} and write $A_i^\circ := A_i \cap M_i^\circ$ with the standard notation $M_i^\circ := \ker(\varphi_i)$. As in the proof of [Ho14, Theorem 3.1], we may and will assume that the elements y and z' are analytic with respect to the modular automorphism group σ^φ and y and $\sigma_{i/2}^\varphi(z')^*$ are finite sums of reduced words w_1, \dots, w_ℓ and w'_1, \dots, w'_ℓ in $A_1 A_2^\circ \cdots A_2^\circ A_1$, respectively. *Unlike usual, we call an element in $M_1 M_2^\circ \cdots M_2^\circ M_1$ a reduced word in what follows.*

Let V be the finite dimensional subspace of M_1 obtained by looking at the letters coming from $A_1^\circ \cup \{1\}$ appearing in y in the same fashion as in the proof of [Ho14, Theorem 3.1]. Namely, V is the linear span of the following A_1 -letters:

- the leftmost A_1 -letters of the reduced words w_i, w_i^* , $1 \leq i \leq \ell$;
- the rightmost A_1 -letters of the reduced words $w'_i, \sigma_{-i}^\varphi(w_i^*)$, $1 \leq i \leq \ell'$;
- the leftmost A_1 -letters of all the reduced words appearing in the elements $w_i^* w_j$, $1 \leq i, j \leq \ell$;
- the rightmost A_1 -letters of all the reduced words appearing in the elements $w'_i \sigma_{-i}^\varphi(w_j^*)$, $1 \leq i, j \leq \ell'$.

Choose an orthonormal basis e_1, \dots, e_m of V with respect to the inner product $(a|b)_{\varphi_1} := \varphi_1(b^*a)$ on M_1 . Denote by W the range of the mapping $a \in M_1 \mapsto a - \sum_{i=1}^m (a|e_i)_{\varphi_1} e_i \in M_1$. It follows that $M_1 = V + W$ is an orthogonal decomposition with respect to the inner product $(\cdot|\cdot)_{\varphi_1}$ defined on M_1 as above.

Let \mathfrak{H} be the direct sum of all the alternating tensor products in $L^2(M_1)^\circ$ and $L^2(M_2)^\circ$ starting and ending with $L^2(M_2)^\circ$. Here $L^2(M_i)^\circ$ denotes the orthogonal complement of the canonical representing vector $\xi_{\varphi_i} \in \mathfrak{P}^{M_i}$ of the given state φ_i . Thanks to $*\text{-alg}(M_1, M_2) = M_1 + \text{span}(M_1 M_2^\circ \cdots M_2^\circ M_1)$ together with the formula of modular conjugation (see [Ue98a, Proposition II-C]), the standard Hilbert space $L^2(M)$ is naturally identified with $L^2(M_1) \oplus L^2(M_1) \otimes \mathfrak{H} \otimes L^2(M_1)$ as M_1 - M_1 -bimodules. Decompose $L^2(M_1) \otimes \mathfrak{H} \otimes L^2(M_1)$ into three subspaces $\mathcal{K}_1, \mathcal{K}_2, \mathcal{L}$ defined by

$$\begin{aligned} \mathcal{K}_1 &:= (V \xi_{\varphi_1}) \otimes \mathfrak{H} \otimes L^2(M_1), \\ \mathcal{K}_2 &:= (\overline{W \xi_{\varphi_1}}) \otimes \mathfrak{H} \otimes (V \xi_{\varphi_1}), \\ \mathcal{L} &:= (\overline{W \xi_{\varphi_1}}) \otimes \mathfrak{H} \otimes (\overline{W \xi_{\varphi_1}}). \end{aligned}$$

It is clear that these subspaces are generated by

$$\begin{aligned} &VM_2^\circ \cdots M_2^\circ M_1 \xi_\varphi, \\ &WM_2^\circ \cdots M_2^\circ V \xi_\varphi, \\ &WM_2^\circ \cdots M_2^\circ W \xi_\varphi, \end{aligned}$$

respectively, in $L^2(M)$, where $\xi_\varphi \in \mathfrak{P}^M$ is the canonical representing vector of the free product state φ . Remark that the direct summand $L^2(M_1)$ in $L^2(M)$ is given by $\overline{M_1 \xi_\psi} = \overline{M_1 \xi_\psi}$ thanks to $\psi \circ E_{M_1} = \psi$ (see e.g. [Ko88, Appendix I]).

Let $\delta > 0$ be arbitrarily chosen. By Lemma 2.1, choose a faithful state $\phi_1 \in Q_*$ such that $\|\psi|_Q - \phi_1\| < \delta$ and Q^{ϕ_1} is diffuse. Denote by $E_Q^{M_1} : M_1 \rightarrow Q$ the unique ψ -preserving conditional expectation and put $\phi := \phi_1 \circ E_Q^{M_1} \circ E_{M_1}$. Then we have $\phi = \phi \circ E_{M_1}$, Q^ϕ is diffuse and $\|\psi - \phi\| = \|\psi|_Q - \phi_1\| < \delta$ so that the canonical representing vectors $\xi_\psi, \xi_\phi \in \mathfrak{P}^M$ of the states ψ, ϕ satisfy $\|\xi_\psi - \xi_\phi\|_{L^2(M)} < \delta^{1/2}$ by the Araki-Powers-Størmer inequality (see [Ta03, Theorem IX.1.2 (iv)]). In what follows, we denote by $P_{\mathcal{X}}$ the orthogonal projection from $L^2(M)$ onto a (closed) subspace \mathcal{X} .

Let $(x_n)_n \in \mathcal{M}^\omega(M)$ such that $x = (x_n)^\omega$ with $C := \sup_n \|x_n\|_\infty$. Then for all $n \in \mathbf{N}$ and all $i \in \{1, 2\}$, we have

$$(3.1) \quad \|P_{\mathcal{K}_i} x_n \xi_\psi\|_{L^2(M)} < C \delta^{1/2} + \|P_{\mathcal{K}_i} x_n \xi_\phi\|_{L^2(M)}.$$

For a while, we will be working with $\|P_{\mathcal{K}_i} x_n \xi_\phi\|_{L^2(M)}$ by the same method used in the proof of [Ho14, Theorem 3.1]. Since Q^ϕ is diffuse, we can choose a unitary $u \in \mathcal{U}(Q^\phi)$ such that $\lim_{k \rightarrow \pm\infty} u^k = 0$ σ -weakly. Consider the unitary transformation $T : L^2(M) \rightarrow L^2(M) : \xi \rightarrow u J^M u J^M \xi =: u \cdot \xi \cdot u^*$. Observe that since $u \in \mathcal{U}(M^\phi)$ and hence $[u, \xi_\phi] = 0$, for all $n \in \mathbf{N}$, all $i \in \{1, 2\}$ and all $k \in \mathbf{Z}$, we have

$$(3.2) \quad T^k P_{\mathcal{K}_i} x_n \xi_\phi = u^k \cdot (P_{\mathcal{K}_i} x_n \xi_\phi) \cdot u^{-k} = P_{u^k \cdot \mathcal{K}_i \cdot u^{-k}} u^k x_n u^{-k} \xi_\phi = P_{T^k \mathcal{K}_i} u^k x_n u^{-k} \xi_\phi.$$

Here is a simple claim, which is just a reconstruction of Claim 1 of [Ho14, §3].

Claim. For any $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for all $i \in \{1, 2\}$, all $\xi, \eta \in \mathcal{K}_i$ and all $k \geq k_0$, we have $|\langle T^k \xi, \eta \rangle_{L^2(M)}| \leq \varepsilon \|\xi\|_{L^2(M)} \|\eta\|_{L^2(M)}$, that is, $T^k \mathcal{K}_i \perp_\varepsilon \mathcal{K}_i$ in the sense of [Ho12a, Definition 2.1].

Proof of the Claim. Denote by J^{M_1} the modular conjugation on $L^2(M_1)$. For $\xi = \sum_{i=1}^m (e_i \xi_{\varphi_1}) \otimes \xi_i, \eta = \sum_{j=1}^m (e_j \xi_{\varphi_1}) \otimes \eta_j \in \mathcal{K}_1$ inside $L^2(M_1) \otimes (\mathfrak{H} \otimes L^2(M_1))$, we have

$$\begin{aligned} |\langle T^k \xi, \eta \rangle_{L^2(M)}| &\leq \sum_{i,j=1}^m |(u^k e_i | e_j)_{\varphi_1}| \|\xi_i\|_{L^2(M)} \|\eta_j\|_{L^2(M)} \\ &\leq \max_{1 \leq i,j \leq m} |(u^k e_i | e_j)_{\varphi_1}| \times \|\xi\|_{L^2(M)} \|\eta\|_{L^2(M)}. \end{aligned}$$

Similarly, for $\xi' = \sum_{i=1}^m \xi'_i \otimes (e_i \xi_{\varphi_1}), \eta' = \sum_{j=1}^m \eta'_j \otimes (e_j \xi_{\varphi_1}) \in \mathcal{K}_2$ inside $(L^2(M_1) \otimes \mathfrak{H}) \otimes L^2(M_1)$, we have

$$\begin{aligned} |\langle T^k \xi', \eta' \rangle_{L^2(M)}| &\leq \sum_{i,j=1}^m \|\xi'_i\|_{L^2(M)} \|\eta'_j\|_{L^2(M)} |\langle u^{-k} J^{M_1} e_j \xi_{\varphi_1}, J^{M_1} e_i \xi_{\varphi_1} \rangle_{L^2(M_1)}| \\ &\leq \max_{1 \leq i,j \leq m} |\langle u^{-k} J^{M_1} e_j \xi_{\varphi_1}, J^{M_1} e_i \xi_{\varphi_1} \rangle_{L^2(M_1)}| \times \|\xi'\|_{L^2(M)} \|\eta'\|_{L^2(M)}. \end{aligned}$$

These two facts together with $\lim_{k \rightarrow \pm\infty} u^k = 0$ σ -weakly imply the desired assertion. \square

Combining Equation (3.2) with the parallelogram law, for all $n \in \mathbb{N}$, all $i \in \{1, 2\}$ and all $k \in \mathbb{Z}$, we have

$$\begin{aligned} \|P_{\mathcal{K}_i} x_n \xi_\phi\|_{L^2(M)}^2 &= \|T^k P_{\mathcal{K}_i} x_n \xi_\phi\|_{L^2(M)}^2 \\ &\leq 2\|(u^k x_n u^{-k} - x_n) \xi_\phi\|_{L^2(M)}^2 + 2\|P_{T^k \mathcal{K}_i} x_n \xi_\phi\|_{L^2(M)}^2. \end{aligned}$$

Thanks to this and the above Claim and since $x \in Q' \cap M^\omega$, the ε -orthogonality technique from [Ho12a, Proposition 2.3] works to show that $\lim_{n \rightarrow \omega} \|P_{\mathcal{K}_i} x_n \xi_\phi\|_{L^2(M)} = 0$ in the same way as in the proof of Claim 2 in [Ho14, §3]. Consequently, we have $\lim_{n \rightarrow \omega} \|P_{\mathcal{K}_i} x_n \xi_\psi\|_{L^2(M)} \leq C\delta^{1/2}$. Since $\delta > 0$ can be arbitrarily small, we finally obtain

$$(3.3) \quad \lim_{n \rightarrow \omega} \|P_{\mathcal{K}_i} x_n \xi_\psi\|_{L^2(M)} = 0, \forall i \in \{1, 2\}.$$

It is standard, see [AH12, Theorem 3.7], that $L^2(M^\omega)$ is embedded into the ultraproduct Hilbert space $L^2(M)^\omega$ by $(a_n)^\omega \xi_{\varphi^\omega} \mapsto (a_n \xi_\varphi)_\omega$ for $(a_n)^\omega \in M^\omega$ with representing sequence $(a_n)_n \in \mathcal{M}^\omega(M)$. Remark that the other mapping $(a_n)^\omega \xi_{\psi^\omega} \mapsto (a_n \xi_\psi)_\omega$ gives exactly the same embedding since we already fix the choice (or realization) of standard forms. By (3.3) together with [AH12, Proposition 3.15, Corollary 3.27, Corollary 3.28], we obtain

$$\begin{aligned} y(x - E_{M_1^\omega}(x)) \xi_{\psi^\omega} &= (y P_{\mathcal{L}} x_n \xi_\psi)_\omega, \\ (y E_{M_1^\omega}(x) - J^{M^\omega} z' J^{M^\omega} E_{M_1^\omega}(x)) \xi_{\psi^\omega} &= ((y E_{M_1}(x_n) - J^M z' J^M E_{M_1}(x_n)) \xi_\psi)_\omega, \\ J^{M^\omega} z' J^{M^\omega} (E_{M_1^\omega}(x) - x) \xi_{\psi^\omega} &= (-J^M z' J^M P_{\mathcal{L}} x_n \xi_\psi)_\omega \end{aligned}$$

inside $L^2(M)^\omega$. Note that $y P_{\mathcal{L}} x_n \xi_\psi$ sits in the closed linear span of $w_i W M_2^\circ \cdots M_2^\circ W \xi_\varphi$, $1 \leq i \leq \ell$, and $J^M z' J^M P_{\mathcal{L}} x_n \xi_\psi$ sits in the closed linear span of $W M_2^\circ \cdots M_2^\circ W w'_j \xi_\varphi$, $1 \leq j \leq \ell'$. Moreover, note that $(y E_{M_1}(x_n) - J^M z' J^M E_{M_1}(x_n)) \xi_\psi \in (y + J^M z' J^M) \overline{M_1 \xi_\psi} = (y + J^M z' J^M) \overline{M_1 \xi_\varphi}$ (n.b. $\psi = \psi \circ E_{M_1}$) as well as that $J^M z' J^M b \xi_\varphi = b \sigma_{i/2}^\varphi(z')^* \xi_\varphi$ for every $b \in M_1$ by [Ta03, Lemma VIII.3.18 (ii)]. This shows that $(y E_{M_1}(x_n) - J^M z' J^M E_{M_1}(x_n)) \xi_\psi$ sits in the closed linear span of $(w_i M_1 + M_1 w'_j) \xi_\varphi$, $1 \leq i \leq \ell$ and $1 \leq j \leq \ell'$.

Observe that the choice of V makes the subspaces $w_i W M_2^\circ \cdots M_2^\circ W \xi_\varphi$, $W M_2^\circ \cdots M_2^\circ W w'_j \xi_\varphi$, $(w_i M_1 + M_1 w'_j) \xi_\varphi$ mutually orthogonal for all $1 \leq i \leq \ell$ and all $1 \leq j \leq \ell'$. This can

easily be checked exactly in the same way as in Claim 3 of [Ho14, §3] (which looks complicated but not difficult). Therefore, $y(x - E_{M_1^\omega}(x))\xi_{\psi^\omega}$, $(yE_{M_1^\omega}(x) - J^{M^\omega}z'J^{M^\omega}E_{M_1^\omega}(x))\xi_{\psi^\omega}$ and $J^{M^\omega}z'J^{M^\omega}(E_{M_1^\omega}(x) - x)\xi_{\psi^\omega}$ are mutually orthogonal in $L^2(M^\omega)$. This finishes the proof of Theorem 3.1. \square

4. PROOFS OF THEOREM A AND COROLLARY B

A key deformation/rigidity result for semifinite von Neumann algebras. Theorem 4.1 below relies on Popa's deformation/rigidity theory [Po01, Po03, Po06] and is an adaptation of Peterson's L^2 -rigidity results [Pe06, Theorems 4.3 and 4.5] for *semifinite* von Neumann algebras using Popa's malleable deformations instead of Peterson's L^2 -derivations.

Recall from [Po03, Po06] that for any inclusion $\mathcal{M} \subset \widetilde{\mathcal{M}}$ of semifinite von Neumann algebras with trace preserving conditional expectation, a trace preserving action $\mathbf{R} \rightarrow \text{Aut}(\widetilde{\mathcal{M}}) : t \mapsto \theta_t$ is called a *malleable deformation* if there exists a period two trace preserving $*$ -automorphism $\beta \in \text{Aut}(\widetilde{\mathcal{M}})$ such that $\beta \circ \theta_t = \theta_{-t} \circ \beta$ for all $t \in \mathbf{R}$. Denote by $E_{\mathcal{M}} : \widetilde{\mathcal{M}} \rightarrow \mathcal{M}$ the unique trace preserving conditional expectation. We will simply denote by $\|\cdot\|_2$ the L^2 -norm associated with the ambient faithful normal semifinite trace. By [Po06, Lemma 2.1], any malleable deformation automatically satisfies the following *transversality property*:

$$\|x - \theta_{2t}(x)\|_2 \leq 2\|\theta_t(x) - E_{\mathcal{M}}(\theta_t(x))\|_2, \forall x \in \mathcal{M} \cap L^2(\mathcal{M}, \text{Tr}).$$

The main result of this subsection is the following uniform convergence theorem for malleable deformations.

Theorem 4.1. *Let $\mathcal{B} \subset \mathcal{M} \subset \widetilde{\mathcal{M}}$ be an inclusion of semifinite von Neumann algebras with trace preserving conditional expectations. Let $\mathbf{R} \rightarrow \text{Aut}(\widetilde{\mathcal{M}}) : t \mapsto \theta_t$ be a trace preserving malleable deformation. Let $p \in \mathcal{M}$ be any nonzero finite trace projection and $\mathcal{Q} \subset p\mathcal{M}p$ any von Neumann subalgebra. Assume that the following conditions hold:*

- (i) *The $p\mathcal{M}p$ - $p\mathcal{M}p$ -bimodule $L^2(p\widetilde{\mathcal{M}}p) \ominus L^2(p\mathcal{M}p)$ is weakly contained in the coarse $p\mathcal{M}p$ - $p\mathcal{M}p$ -bimodule $L^2(p\mathcal{M}p) \otimes L^2(p\mathcal{M}p)$.*
- (ii) *The von Neumann algebra \mathcal{Q} has no amenable direct summand.*
- (iii) *There exists a nonprincipal ultrafilter $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$ such that $\mathcal{Q}' \cap (p\mathcal{M}p)^\omega \not\subseteq_{\mathcal{M}^\omega} \mathcal{B}^\omega$.*
- (iv) *Denote by $E_{\mathcal{B}} : \mathcal{M} \rightarrow \mathcal{B}$ the unique trace preserving conditional expectation. For every net $(v_i)_{i \in I}$ of unitaries in $\mathcal{U}(p\mathcal{M}p)$ satisfying $\lim_i \|E_{\mathcal{B}}(b^*v_i a)\|_2 = 0$ for all $a, b \in p\mathcal{M}$, we have $\lim_i \|E_{\mathcal{M}}(d^*v_i c)\|_2 = 0$ for all $c, d \in p(\widetilde{\mathcal{M}} \ominus \mathcal{M})$.*

Then the map $\mathbf{R} \rightarrow \text{Aut}(\widetilde{\mathcal{M}}) : t \mapsto \theta_t$ converges uniformly on $\text{Ball}(\mathcal{Q})$ in $\|\cdot\|_2$ as $t \rightarrow 0$.

Proof. Put $\mathcal{P} = \mathcal{Q}' \cap (p\mathcal{M}p)^\omega$. For every $t \in \mathbf{R}$, define $\theta_t^\omega \in \text{Aut}(\widetilde{\mathcal{M}}^\omega)$ by $\theta_t^\omega((x_n)^\omega) = (\theta_t(x_n))^\omega$. We note that the map $\mathbf{R} \rightarrow \text{Aut}(\widetilde{\mathcal{M}}^\omega) : t \mapsto \theta_t^\omega$ need not be continuous. However, exploiting Popa's spectral gap argument [Po06], we can show the following uniform convergence result.

Claim. *The map $\mathbf{R} \rightarrow \text{Aut}(\widetilde{\mathcal{M}}^\omega) : t \mapsto \theta_t^\omega$ converges uniformly on $\text{Ball}(\mathcal{P})$ in $\|\cdot\|_2$ as $t \rightarrow 0$.*

Proof of Claim. For the Claim, we will only use Conditions (i),(ii). Assume by contradiction that the map $\mathbf{R} \rightarrow \text{Aut}(\widetilde{\mathcal{M}}^\omega) : t \mapsto \theta_t^\omega$ does not converge uniformly on $\text{Ball}(\mathcal{P})$ in $\|\cdot\|_2$ as $t \rightarrow 0$. Thus there exist $c > 0$, a sequence $(t_k)_k$ of positive reals such that $\lim_k t_k = 0$ and a sequence $(X_k)_k$ in $\text{Ball}(\mathcal{P})$ such that $\|X_k - \theta_{2t_k}^\omega(X_k)\|_2 \geq 2c$ for all $k \in \mathbf{N}$. Write $X_k = (x_n^{(k)})^\omega$ with $x_n^{(k)} \in \text{Ball}(p\mathcal{M}p)$ satisfying $\lim_{n \rightarrow \omega} \|yx_n^{(k)} - x_n^{(k)}y\|_2 = 0$ and $2c \leq \|X_k - \theta_{2t_k}^\omega(X_k)\|_2 = \lim_{n \rightarrow \omega} \|x_n^{(k)} - \theta_{2t_k}(x_n^{(k)})\|_2$ for all $k \in \mathbf{N}$ and all $y \in \mathcal{Q}$.

Denote by I the directed set of all pairs $(\mathcal{F}, \varepsilon)$ with $\mathcal{F} \subset \text{Ball}(\mathcal{Q})$ finite subset and $\varepsilon > 0$. Let $i = (\mathcal{F}, \varepsilon) \in I$ and put $\delta = \min(\frac{\varepsilon}{6}, \frac{\varepsilon}{8})$. Choose $k \in \mathbf{N}$ large enough so that $\|p - \theta_{t_k}(p)\|_2 \leq \delta$ and $\|a - \theta_{t_k}(a)\|_2 \leq \varepsilon/6$ for all $a \in \mathcal{F}$. Then choose $n \in \mathbf{N}$ large enough so that $\|x_n^{(k)} - \theta_{2t_k}(x_n^{(k)})\|_2 \geq c$ and $\|ax_n^{(k)} - x_n^{(k)}a\|_2 \leq \varepsilon/3$ for all $a \in \mathcal{F}$.

Put $\xi_i = \theta_{t_k}(x_n^{(k)}) - E_{\mathcal{M}}(\theta_{t_k}(x_n^{(k)})) \in L^2(\widetilde{\mathcal{M}}) \ominus L^2(\mathcal{M})$ and $\eta_i = p\xi_i p \in L^2(p\widetilde{\mathcal{M}}p) \ominus L^2(p\mathcal{M}p)$. By the transversality property of the malleable deformation (θ_t) , we have

$$\|\xi_i\|_2 \geq \frac{1}{2}\|x_n^{(k)} - \theta_{2t_k}(x_n^{(k)})\|_2 \geq \frac{c}{2}.$$

Observe that $\|p\theta_{t_k}(x_n^{(k)})p - \theta_{t_k}(x_n^{(k)})\|_2 \leq 2\|p - \theta_{t_k}(p)\|_2 \leq 2\delta$. Since $p \in \mathcal{M}$, by Pythagoras theorem, we moreover have

$$\|p\theta_{t_k}(x_n^{(k)})p - \theta_{t_k}(x_n^{(k)})\|_2^2 = \|E_{\mathcal{M}}(p\theta_{t_k}(x_n^{(k)})p - \theta_{t_k}(x_n^{(k)}))\|_2^2 + \|\eta_i - \xi_i\|_2^2$$

and hence $\|\eta_i - \xi_i\|_2 \leq 2\delta$. This implies that

$$\|\eta_i\|_2 \geq \|\xi_i\|_2 - \|\eta_i - \xi_i\|_2 \geq \frac{c}{2} - 2\delta \geq \frac{c}{4}.$$

For all $x \in p\mathcal{M}p$, we have

$$\|x\eta_i\|_2 = \|(1 - E_{\mathcal{M}})(x\theta_{t_k}(x_n^{(k)})p)\|_2 \leq \|x\theta_{t_k}(x_n^{(k)})p\|_2 \leq \|x\|_2.$$

By Popa's spectral gap argument [Po06], for all $a \in \mathcal{F} \subset \text{Ball}(p\mathcal{M}p)$, we have

$$\begin{aligned} \|a\eta_i - \eta_i a\|_2 &= \|(1 - E_{\mathcal{M}})(a\theta_{t_k}(x_n^{(k)})p - p\theta_{t_k}(x_n^{(k)})a)\|_2 \\ &\leq \|a\theta_{t_k}(x_n^{(k)})p - p\theta_{t_k}(x_n^{(k)})a\|_2 \\ &\leq 2\|a - \theta_{t_k}(a)\|_2 + 2\|p - \theta_{t_k}(p)\|_2 + \|ax_n^{(k)} - x_n^{(k)}a\|_2 \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence $\eta_i \in L^2(p\widetilde{\mathcal{M}}p) \ominus L^2(p\mathcal{M}p)$ is a net of vectors satisfying $\limsup_i \|x\eta_i\|_2 \leq \|x\|_2$ for all $x \in p\mathcal{M}p$, $\liminf_i \|\eta_i\|_2 \geq \frac{c}{4}$ and $\lim_i \|a\eta_i - \eta_i a\|_2 = 0$ for all $a \in \mathcal{Q}$. By Condition (i), it follows that $\mathcal{Q} \subset p\mathcal{M}p$ has an amenable direct summand by Connes's characterization of amenability [Co75] for finite von Neumann algebras (see also [Io12, Lemma 2.3]). This is a contradiction to Condition (ii) and finishes the proof of the Claim. \square

Next, we use an idea due to Peterson [Pe06] in combination with the above Claim to bring down the uniform convergence to $\text{Ball}(\mathcal{Q})$. In what follows, we will use Conditions (iii), (iv). Let $\varepsilon > 0$. By the above Claim, there exists $t_0 > 0$ such that $\|v - \theta_t^\omega(v)\|_2 < \frac{\varepsilon^2}{16}$ for all $v \in \mathcal{U}(\mathcal{P})$ (recall $\mathcal{P} = \mathcal{Q}' \cap (p\mathcal{M}p)^\omega$) and all $t \in [-t_0, t_0]$. Fix $x \in \text{Ball}(\mathcal{Q})$ and $t \in [-t_0, t_0]$. We will show that $\|x - \theta_{2t}(x)\|_2 \leq \varepsilon$.

Denote by I the directed set of all pairs (\mathcal{F}, δ) with $\mathcal{F} \subset \text{Ball}(p\mathcal{M})$ finite subset and $\delta > 0$. Fix $i = (\mathcal{F}, \delta) \in I$. By Condition (iii), we have that $\mathcal{P} \not\prec_{\mathcal{M}^\omega} \mathcal{B}^\omega$. This implies, in particular, that there exists a unitary $u \in \mathcal{U}(\mathcal{P})$ such that $\|E_{\mathcal{B}^\omega}(b^*ua)\|_2 < \delta$ for all $a, b \in \mathcal{F}$. Since $p\mathcal{M}p$ is a finite von Neumann algebra, we may write $u = (u_n)^\omega \in \mathcal{U}(\mathcal{P})$ for some $(u_n)_n \in \ell^\infty(\mathbf{N}, p\mathcal{M}p)$ such that $u_n \in \mathcal{U}(p\mathcal{M}p)$ for all $n \in \mathbf{N}$. Observe that $\lim_{n \rightarrow \omega} \|u_n x - x u_n\|_2 = \|u x - x u\|_2 = 0$, $\|E_{\mathcal{B}^\omega}(b^*ua)\|_2 = \lim_{n \rightarrow \omega} \|E_{\mathcal{B}}(b^*u_n a)\|_2 < \delta$ for all $a, b \in \mathcal{F}$ and $\|u - \theta_t^\omega(u)\|_2 = \lim_{n \rightarrow \omega} \|u_n - \theta_t(u_n)\|_2$. Thus, there exists $n \in \mathbf{N}$ large enough such that $v_i := u_n \in \mathcal{U}(p\mathcal{M}p)$ satisfies the following properties:

- $\|v_i x - x v_i\|_2 \leq \delta$,
- $\|E_{\mathcal{B}}(b^*v_i a)\|_2 \leq \delta$ for all $a, b \in \mathcal{F}$ and
- $\|v_i - \theta_t(v_i)\|_2 \leq \|u - \theta_t^\omega(u)\|_2 + \frac{\varepsilon^2}{16} \leq \frac{\varepsilon^2}{8}$.

Put $\delta_t(y) = \theta_t(y) - E_{\mathcal{M}}(\theta_t(y)) \in \widetilde{\mathcal{M}} \ominus \mathcal{M}$ for all $y \in p\mathcal{M}p$. For all $i \in I$, we have

$$\begin{aligned}
 (4.1) \quad \|\delta_t(x)\|_2^2 &= \langle \delta_t(x), \delta_t(x) \rangle \leq |\langle \delta_t(v_i x v_i^*), \delta_t(x) \rangle| + \|v_i x v_i^* - x\|_2 \\
 &\leq |\langle v_i \delta_t(x) v_i^*, \delta_t(x) \rangle| + \|v_i x v_i^* - x\|_2 + 2\|v_i - \theta_t(v_i)\|_2 \\
 &\leq |\langle v_i \delta_t(x) v_i^*, \delta_t(x) \rangle| + \|v_i x v_i^* - x\|_2 + \frac{\varepsilon^2}{4}.
 \end{aligned}$$

Since $\lim_i \|E_{\mathcal{B}}(b^* v_i a)\|_2 = 0$ for all $a, b \in p\mathcal{M}$, we have $\lim_i \|E_{\mathcal{M}}(d^* v_i c)\|_2 = 0$ for all $c, d \in p(\widetilde{\mathcal{M}} \ominus \mathcal{M})$ by Condition (iv). In particular, using Cauchy-Schwarz inequality in $L^2(\widetilde{\mathcal{M}})$, we have

$$\begin{aligned}
 (4.2) \quad \limsup_i |\langle v_i \delta_t(x) v_i^*, \delta_t(x) \rangle| &= \limsup_i |\langle \delta_t(x)^* v_i \delta_t(x), v_i \rangle| \\
 &= \limsup_i |\langle E_{\mathcal{M}}(\delta_t(x)^* v_i \delta_t(x)), v_i \rangle| \\
 &\leq \limsup_i \|E_{\mathcal{M}}((p\delta_t(x))^* v_i p\delta_t(x))\|_2 \|v_i\|_2 \\
 &= 0.
 \end{aligned}$$

Combining (4.1) and (4.2) with the first property of the net $(v_i)_{i \in I}$ and the transversality property of the malleable deformation (θ_t) , we obtain

$$\|x - \theta_{2t}(x)\|_2 \leq 2\|\delta_t(x)\|_2 \leq \varepsilon.$$

Since the above inequality holds for all $x \in \text{Ball}(\mathcal{Q})$ and all $t \in [-t_0, t_0]$, we have obtained that the map $\mathbf{R} \rightarrow \text{Aut}(\widetilde{\mathcal{M}}) : t \mapsto \theta_t$ converges uniformly on $\text{Ball}(\mathcal{Q})$ in $\|\cdot\|_2$ as $t \rightarrow 0$. This finishes the proof of Theorem 4.1. \square

As a corollary to Theorem 4.1, we obtain the following ‘location’ result for subalgebras in semifinite amalgamated free product von Neumann algebras. For each $i \in \{1, 2\}$, let $\mathcal{B} \subset \mathcal{M}_i$ be an inclusion of σ -finite semifinite von Neumann algebras with expectation $E_i : \mathcal{M}_i \rightarrow \mathcal{B}$. Let $\text{Tr}_{\mathcal{B}}$ be a faithful normal semifinite trace such that the weight $\text{Tr}_{\mathcal{B}} \circ E_i$ is tracial on \mathcal{M}_i for all $i \in \{1, 2\}$. Then the amalgamated free product $(\mathcal{M}, E) = (\mathcal{M}_1, E_1) *_{\mathcal{B}} (\mathcal{M}_2, E_2)$ is semifinite and the weight $\text{Tr} = \text{Tr}_{\mathcal{B}} \circ E$ is tracial on \mathcal{M} as remarked in Section 2.

Corollary 4.2. *Keep the same notation as above. Assume moreover that \mathcal{B} is amenable. Let $p \in \mathcal{M}$ be any nonzero finite trace projection and $\mathcal{Q} \subset p\mathcal{M}p$ any von Neumann subalgebra with no amenable direct summand such that $\mathcal{Q}' \cap (p\mathcal{M}p)^{\omega} \not\preceq_{\mathcal{M}^{\omega}} \mathcal{B}^{\omega}$ for some nonprincipal ultrafilter $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$.*

Then for every nonzero projection $z \in \mathcal{Q}' \cap p\mathcal{M}p$, there exists $i \in \{1, 2\}$ such that $\mathcal{Q}z \preceq_{\mathcal{M}} \mathcal{M}_i$.

Proof. Put

$$\widetilde{\mathcal{M}} = \mathcal{M} *_{\mathcal{B}} (\mathcal{B} \overline{\otimes} L(\mathbf{F}_2))$$

and consider the trace preserving free malleable deformation (θ_t) from [IPP05, Section 2] on $\widetilde{\mathcal{M}}$ (see [BHR12, Section 3] for further details).

We now check that we can apply Theorem 4.1 to our situation.

- (i) Since \mathcal{B} is amenable, the $p\mathcal{M}p$ - $p\mathcal{M}p$ -bimodule $L^2(p\widetilde{\mathcal{M}}p) \ominus L^2(p\mathcal{M}p)$ is weakly contained in the coarse $p\mathcal{M}p$ - $p\mathcal{M}p$ -bimodule $L^2(p\mathcal{M}p) \otimes L^2(p\mathcal{M}p)$ (see e.g. the proof of [CH08, Proposition 3.1]).
- (ii) By assumption, the von Neumann algebra \mathcal{Q} has no amenable direct summand.
- (iii) By assumption, we have $\mathcal{Q}' \cap (p\mathcal{M}p)^{\omega} \not\preceq_{\mathcal{M}^{\omega}} \mathcal{B}^{\omega}$ for some nonprincipal ultrafilter $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$.

- (iv) Let $(v_i)_{i \in I}$ be any net of unitaries in $\mathcal{U}(p\mathcal{M}p)$ such that $\lim_i \|E_{\mathcal{B}}(b^*v_i a)\|_2 = 0$ for all $a, b \in p\mathcal{M}$. Since $\widetilde{\mathcal{M}} = \mathcal{M} *_B (\mathcal{B} \overline{\otimes} L(\mathbf{F}_2))$, the proof of [BHR12, Theorem 2.5, Claim] implies that $\lim_i \|E_{\mathcal{M}}(d^*v_i c)\|_2 = 0$ for all $c, d \in p(\widetilde{\mathcal{M}} \ominus \mathcal{M})$.

Therefore, Theorem 4.1 implies that the map $\mathbf{R} \rightarrow \text{Aut}(\widetilde{\mathcal{M}}) : t \mapsto \theta_t$ converges uniformly on $\text{Ball}(\mathcal{Q})$ in $\|\cdot\|_2$ as $t \rightarrow 0$. Fix now any nonzero projection $z \in \mathcal{Q}' \cap p\mathcal{M}p$. We still have that the map $\mathbf{R} \rightarrow \text{Aut}(\widetilde{\mathcal{M}}) : t \mapsto \theta_t$ converges uniformly on $\text{Ball}(\mathcal{Q}z)$ in $\|\cdot\|_2$ as $t \rightarrow 0$. Then, [BHR12, Theorem 3.3] implies that there exists $i \in \{1, 2\}$ such that $\mathcal{Q}z \preceq_{\mathcal{M}} \mathcal{M}_i$. \square

Proof of Theorem A. Theorem A will be a consequence of the following optimal result that generalizes [Ho14, Theorem D] to arbitrary free product von Neumann algebras.

Theorem 4.3. *For each $i \in \{1, 2\}$, let (M_i, φ_i) be any σ -finite von Neumann algebra endowed with a faithful normal state. Denote by $(M, \varphi) = (M_1, \varphi_1) * (M_2, \varphi_2)$ the free product. Let $Q \subset M$ be any von Neumann subalgebra with separable predual and with expectation such that $Q \cap M_1$ is diffuse and with expectation. Let $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$ be any nonprincipal ultrafilter on \mathbf{N} .*

Denote by $z \in \mathcal{Z}(Q' \cap M^\omega)$ the unique central projection such that $(Q' \cap M^\omega)z$ is diffuse and $(Q' \cap M^\omega)z^\perp$ is atomic. Then the following conditions hold:

- $z \in \mathcal{Z}(Q' \cap M) = \mathcal{Z}(Q' \cap M_1)$,
- $Qz \subset zM_1z$ and
- $(Q' \cap M^\omega)z^\perp = (Q' \cap M)z^\perp = (Q' \cap M_1)z^\perp$.

Throughout the rest of this section, let $(M, \varphi) = (M_1, \varphi_1) * (M_2, \varphi_2)$ be as in Theorem 4.3. Observe that M_1 is diffuse by assumption. Proposition 2.7 (1) implies that $(M_1)' \cap M \subset M_1$. Therefore, there exists a unique faithful normal conditional expectation $E_{M_1} : M \rightarrow M_1$ by [Co72, Théorème 1.5.5]. We fix a nonprincipal ultrafilter $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$.

For Lemmas 4.4 and 4.5 below, we moreover fix a faithful state $\psi \in M_*$ such that $\psi = \psi \circ E_{M_1}$. Whenever $q \in M^\psi$ is a nonzero projection, put $\psi_q = \frac{\psi(q \cdot q)}{\psi(q)} \in (qMq)_*$.

Lemma 4.4. *Let $q \in (M_1)^\psi$ be any nonzero projection and $Q \subset qMq$ any non type I subfactor with separable predual that is amenable and globally invariant under the modular automorphism group σ^{ψ_q} and such that $Q \cap qM_1q$ is diffuse. Then $Q \subset qM_1q$.*

Proof. The proof of Lemma 4.4 is inspired by the one of [Ho12b, Theorem 8.1]. We will consider successively the cases when Q is of type II_1 , of type II_∞ and of type III .

Case type II_1 . Assume that $q \in (M_1)^\psi$ is any nonzero projection and $Q \subset qMq$ is any type II_1 subfactor with separable predual that is amenable and globally invariant under the modular automorphism group σ^{ψ_q} and such that $Q \cap qM_1q$ is diffuse. Then we have $Q \subset qM_1q$.

We start by showing the following claim.

Claim. For any nonzero projection $z \in \mathcal{Z}(Q' \cap qMq)$, we have $Qz \preceq_M M_1$.

Proof of the Claim. By contradiction, assume that there exists a nonzero projection $z \in \mathcal{Z}(Q' \cap qMq)$ such that $Qz \not\preceq_M M_1$. Since $Q' \cap qMq \subset qM_1q$ by Proposition 2.7 (1) and $Q' \cap qMq \subset qM_1q$ is globally invariant under the modular automorphism group σ^{ψ_q} , we have $z \in (M_1)^\psi$. Write $Q = \bigvee_{n \in \mathbf{N}} Q_n$ where $(Q_n)_n$ is an increasing sequence of finite dimensional subfactors of Q of the form $Q_n \cong \mathbf{M}_{2^n}(\mathbf{C})$. Since the inclusion

$$(Q'_n \cap Q)z \subset Qz \cong Q'_n \cap Q \subset Q$$

(*n.b.* Q is a factor) has finite index, Lemma 2.3 implies that $(Q'_n \cap Q)z \not\preceq_M M_1$ for all $n \in \mathbf{N}$.

Then for every $n \in \mathbf{N}$, choose a unitary $u_n \in \mathcal{U}((Q'_n \cap Q)z)$ such that $\|E_{M_1}(u_n)\|_\psi \leq \frac{1}{n+1}$. Since Qz is finite with expectation, we have $(u_n)_n \in \mathcal{M}^\omega(zMz)$ and hence we may define $u = (u_n)^\omega \in (zMz)^\omega = zM^\omega z \subset M^\omega$. We then have $u \in (Qz)' \cap (Qz)^\omega$ and $E_{M_1^\omega}(u) = 0$ since

$$\|E_{M_1^\omega}(u)\|_{\psi^\omega} = \lim_{n \rightarrow \omega} \|E_{M_1}(u_n)\|_\psi = 0.$$

Observe that $(Qz \cap zM_1z) \oplus z^\perp M_1 z^\perp \subset M_1$ is a diffuse von Neumann subalgebra that is globally invariant under the modular automorphism group σ^ψ . Since $u \in (Qz)' \cap (Qz)^\omega$, we have $u \in ((Qz \cap zM_1z) \oplus z^\perp M_1 z^\perp)' \cap M^\omega$. For all $n \in \mathbf{N}$, since we moreover have $u u_n = u_n u$ and $u^* u = z$, Theorem 3.1 implies that

$$\begin{aligned} \|E_{M_1}(u_n)u - uE_{M_1}(u_n)\|_{\psi^\omega} &= \|(E_{M_1}(u_n) - u_n)u + u(u_n - E_{M_1}(u_n))\|_{\psi^\omega} \\ &\geq \|u(u_n - E_{M_1}(u_n))\|_{\psi^\omega} \quad (\text{use Theorem 3.1 here}) \\ &\geq \|z\|_\psi - \|E_{M_1}(u_n)\|_\psi. \end{aligned}$$

Observe that $\lim_{n \rightarrow \infty} E_{M_1}(u_n) = 0$ σ -strongly. By taking the limit as $n \rightarrow \infty$ in the above inequality, we obtain $z = 0$, a contradiction. This finishes the proof of the Claim. \square

The set \mathfrak{R} of projections $r \in Q' \cap qMq = Q' \cap qM_1q$ (by Proposition 2.7 (1)) such that $Qr \subset rM_1r$ attains its maximum in a unique projection z that belongs to $\mathcal{Z}(Q' \cap qMq) = \mathcal{Z}(Q' \cap qM_1q)$. (In fact, \mathfrak{R} is invariant under the adjoint action of $\mathcal{U}(Q' \cap qM_1q)$, and $z := \bigvee_{r \in \mathfrak{R}} r \in \mathcal{Z}(Q' \cap qM_1q)$ must satisfy $xz = E_{M_1}(x)z = zE_{M_1}(x)z$ for all $x \in Q$.) Assume by contradiction that $z \neq q$. Put $z^\perp := q - z \in \mathcal{Z}(Q' \cap qMq)$. By assumption, we have $z^\perp \neq 0$.

By the previous Claim, we have that $Qz^\perp \preceq_M M_1$. Then there exist $n \geq 1$, a projection $p \in \mathbf{M}_n(M_1)$, a nonzero partial isometry $v \in (z^\perp M \otimes \mathbf{M}_{1,n}(\mathbf{C}))p$ and a unital normal $*$ -homomorphism $\pi : Qz^\perp \rightarrow p\mathbf{M}_n(M_1)p$ such that the inclusion $\pi(Qz^\perp) \subset p\mathbf{M}_n(M_1)p$ is with expectation (see Theorem 2.2 due to the first named author and Isono [HI15] for this important property) and $av = v\pi(a)$ for all $a \in Qz^\perp$. By Proposition 2.7 (1), we obtain that $v \in (z^\perp M_1 \otimes \mathbf{M}_{1,n}(\mathbf{C}))p$ and hence $vv^* \in z^\perp(Q' \cap qMq)z^\perp = z^\perp(Q' \cap qM_1q)z^\perp$ and $Qz^\perp vv^* \subset vv^* z^\perp M_1 z^\perp vv^*$. Since $vv^* \leq z^\perp$, $vv^* \neq 0$ and $Q(z + vv^*) \subset (z + vv^*)M_1(z + vv^*)$, this contradicts the maximality of $z \in Q' \cap qM_1q$ and finishes the proof in the case when Q is of type II_1 .

Case type II_∞ . Assume that $q \in (M_1)^\psi$ is any nonzero projection and $Q \subset qMq$ is any type II_∞ subfactor with separable predual that is amenable and globally invariant under the modular automorphism group σ^{ψ_q} and such that $Q \cap qM_1q$ is diffuse. Then we have $Q \subset qM_1q$.

Choose a faithful normal semifinite trace Tr on Q and write $\psi_q = \text{Tr}(T \cdot)$ for some positive nonsingular operator $T \in L^1(Q, \text{Tr})_+$ (see e.g. [Ta03, Corollary VIII.3.6, Lemma IX.2.12]). Define the abelian von Neumann subalgebra $B = \{T^{\text{it}} : t \in \mathbf{R}\}'' \subset Q$. Since $\sigma_t^{\psi_q} = \text{Ad}(T^{\text{it}})$ for all $t \in \mathbf{R}$, we have $Q^{\psi_q} = B' \cap Q$. Observe that since the inclusion $Q \cap qM_1q \subset Q$ is globally invariant under the modular automorphism group σ^{ψ_q} , the diffuse von Neumann subalgebra $Q \cap qM_1q \subset Q$ is also semifinite and hence its centralizer $(Q \cap qM_1q)^{\psi_q}$ is diffuse (see e.g. [Ue98b, Lemma 11]). By Proposition 2.7 (1) and since B is abelian, we have

$$B \subset (Q^{\psi_q})' \cap Q^{\psi_q} \subset ((Q \cap qM_1q)^{\psi_q})' \cap Q^{\psi_q} \subset Q^{\psi_q} \cap qM_1q = (Q \cap qM_1q)^{\psi_q}.$$

For every $k \in \mathbf{N}$, we denote by q_k the spectral projection of T for the interval $[\frac{1}{k}, +\infty)$. Then all q_k are Tr -finite projections in B such that $q_k \rightarrow q$, the unit of Q , σ -strongly as $k \rightarrow \infty$. Since $q_k \in (Q \cap qM_1q)^{\psi_q}$, the type II_1 subfactor $q_k Q q_k \subset q_k M q_k$ is amenable and globally invariant under the modular automorphism group $\sigma^{\psi_{q_k}}$ and $q_k Q q_k \cap q_k M_1 q_k = q_k (Q \cap qM_1q) q_k$ is diffuse. We may then apply the result obtained in the first case to the II_1 subfactor $q_k Q q_k \subset q_k M q_k$

and we have that $q_k Q q_k \subset q_k M_1 q_k$ for all $k \in \mathbf{N}$. Since $q_k \rightarrow q$ σ -strongly as $k \rightarrow \infty$, we obtain $Q \subset q M_1 q$. This finishes the proof in the case when Q is of type II_∞ .

Case type III. Assume that $q \in (M_1)^\psi$ is any nonzero projection and $Q \subset q M q$ is any type III subfactor with separable predual that is amenable and globally invariant under the modular automorphism group σ^{ψ_q} and such that $Q \cap q M_1 q$ is diffuse. Then we have $Q \subset q M_1 q$.

By combining results on the classification theory of amenable factors [Co72, Co75, Ha85] together with [FM75, Kr75], there exists a hyperfinite ergodic nonsingular equivalence relation \mathcal{R} defined on a standard probability space (X, μ) such that $Q = L(\mathcal{R})$. Put $A = L^\infty(X)$ and denote by $E_A : Q \rightarrow A$ the unique faithful normal conditional expectation. Denote by $E_Q : q M q \rightarrow Q$ the unique ψ_q -preserving conditional expectation. Choose any faithful state $\tau_A \in A_*$ and put $\phi = \tau_A \circ E_A \circ E_Q \in (q M q)_*$. Observe that $A \subset (q M q)^\phi$ and Q is globally invariant under the modular automorphism group σ^ϕ .

Let $(\mathcal{R}_n)_{n \in \mathbf{N}}$ be an increasing sequence of finite subequivalence relations of \mathcal{R} such that $\mathcal{R} = \bigvee_{n \in \mathbf{N}} \mathcal{R}_n$. Put $Q_n = L(\mathcal{R}_n)$ for all $n \in \mathbf{N}$ so that $Q = \bigvee_{n \in \mathbf{N}} Q_n$. Note that $A \subset Q_n$ is still a Cartan subalgebra and Q_n is globally invariant under the modular automorphism group σ^ϕ for all $n \in \mathbf{N}$ because $\phi|_Q = \tau_A \circ E_A$. Observe that since \mathcal{R}_n is finite, that is, \mathcal{R}_n has finite orbits almost everywhere, Q_n is a countable direct sum of finite type I von Neumann algebras. Therefore using [Ka82, Corollary 3.19], up to conjugating by a unitary in $\mathcal{U}(Q_n)$, the inclusion $A \subset Q_n$ is of the following form:

$$(4.3) \quad (A \subset Q_n) \cong \left(\bigoplus_{k \in \mathbf{N}} \mathcal{Z}_n^{(k)} \otimes \mathbf{C}^{\oplus k} \subset \bigoplus_{k \in \mathbf{N}} \mathcal{Z}_n^{(k)} \otimes \mathbf{M}_k(\mathbf{C}) \right),$$

where $\mathcal{Z}_n^{(k)}$ is a diffuse abelian von Neumann algebra for all $n, k \in \mathbf{N}$.

Claim. For any nonzero projection $z \in \mathcal{Z}(Q' \cap q M q)$, we have $Az \preceq_M M_1$.

Proof of the Claim. By contradiction, assume that there exists a nonzero projection $z \in \mathcal{Z}(Q' \cap q M q)$ such that $Az \not\preceq_M M_1$. Observe that $z \in (q M q)^\phi \cap (M_1)^\psi$. Using the structure of the inclusion $A \subset Q_n$ as in (4.3), we see that the inclusion $Q'_n \cap A \subset A$ is of the form:

$$(4.4) \quad (Q'_n \cap A \subset A) \cong \left(\bigoplus_{k \in \mathbf{N}} \mathcal{Z}_n^{(k)} \otimes \mathbf{C}1 \subset \bigoplus_{k \in \mathbf{N}} \mathcal{Z}_n^{(k)} \otimes \mathbf{C}^{\oplus k} \right).$$

Using (4.4), it follows that the inclusion

$$(Q'_n \cap A)z \subset Az \cong Q'_n \cap A \subset A$$

has essentially finite index and Lemma 2.3 implies that $(Q'_n \cap A)z \not\preceq_M M_1$ for all $n \in \mathbf{N}$. (Remark that this can easily be confirmed directly, since $Q'_n \cap A \subset A$ are commutative.)

Then for every $n \in \mathbf{N}$, choose a unitary $u_n \in \mathcal{U}((Q'_n \cap A)z)$ such that $\|E_{M_1}(u_n)\|_\psi \leq \frac{1}{n+1}$. Since $u_n \in (zMz)^{\phi_z}$ for all $n \in \mathbf{N}$, we may define $u = (u_n)^\omega \in (zMz)^\omega = zM^\omega z \subset M^\omega$. We then have $u \in (Qz)' \cap (Az)^\omega$ and $E_{M_1^\omega}(u) = 0$. Observe that $u \in ((Qz \cap zM_1z) \oplus z^\perp M_1 z^\perp)' \cap M^\omega$. For all $n \in \mathbf{N}$, Theorem 3.1 implies, as in Case type II_1 , that

$$\|E_{M_1}(u_n)u - uE_{M_1}(u_n)\|_{\psi^\omega} \geq \|z\|_\psi - \|E_{M_1}(u_n)\|_\psi$$

and hence $z = 0$, a contradiction. This finishes the proof of the Claim. \square

The set of projections $r \in Q' \cap q M q = Q' \cap q M_1 q$ (by Proposition 2.7 (1)) such that $Qr \subset r M_1 r$ attains its maximum in a unique projection z that belongs to $\mathcal{Z}(Q' \cap q M q) = \mathcal{Z}(Q' \cap q M_1 q)$ (see Case type II_1). Assume by contradiction that $z \neq q$. Put $z^\perp := q - z \in \mathcal{Z}(Q' \cap q M q)$. By assumption, we have $z^\perp \neq 0$ and moreover $z^\perp \in Q' \cap q M q \subset A' \cap q M q$.

By the previous Claim, we have that $Az^\perp \preceq_M M_1$. Then there exist $n \geq 1$, a projection $p \in \mathbf{M}_n(M_1)$, a nonzero partial isometry $v \in (z^\perp M \otimes \mathbf{M}_{1,n}(\mathbf{C}))p$ and a unital normal $*$ -homomorphism $\pi : Az^\perp \rightarrow p\mathbf{M}_n(M_1)p$ such that the inclusion $\pi(Az^\perp) \subset p\mathbf{M}_n(M_1)p$ is with expectation (see Theorem 2.2) and $av = v\pi(a)$ for all $a \in Az^\perp$. Since $z^\perp \in Q' \cap qMq \subset A' \cap qMq$, we may define the unital normal $*$ -homomorphism $\iota : A \rightarrow Az^\perp : a \mapsto az^\perp$. Then $\pi \circ \iota : A \rightarrow p\mathbf{M}_n(M_1)p$ is unital normal $*$ -homomorphism such that the inclusion $(\pi \circ \iota)(A) \subset p\mathbf{M}_n(M_1)p$ is with expectation and $av = \iota(a)v = v\pi(\iota(a)) = v(\pi \circ \iota)(a)$ for all $a \in A$.

Put $N = \mathcal{N}_{qMq}(A)''$ and observe that $Q \subset N$. Since $v^*v \in (\pi \circ \iota)(A)' \cap p\mathbf{M}_n(M)p$ and since $(\pi \circ \iota)(A) \subset p\mathbf{M}_n(M_1)p$ is diffuse and with expectation, we have $v^*v \in (\pi \circ \iota)(A)' \cap p\mathbf{M}_n(M_1)p$ by Proposition 2.7 (2) and hence we may assume that $p = v^*v$. Since the inclusion $A \subset N$ is regular, we moreover have $v^*Nv \subset p\mathbf{M}_n(M_1)p$ by Proposition 2.7 (2).

We have $vv^* \in (Az^\perp)' \cap z^\perp M z^\perp = z^\perp (A' \cap qMq) z^\perp \subset z^\perp N z^\perp$. Since the inclusion $Q \subset N$ is with expectation (because so is $Q \subset qMq$) and since Q is of type III, it follows that N is also of type III (see [Ta02, Lemma V.2.29]) and so is $z^\perp N z^\perp$. If we denote by $r \in \mathcal{Z}(z^\perp N z^\perp)$ the central support in $z^\perp N z^\perp$ of the projection $vv^* \in z^\perp N z^\perp$, we have $vv^* \sim r$ in $z^\perp N z^\perp$. There exists a partial isometry $u \in z^\perp N z^\perp$ such that $u^*u = vv^*$ and $uu^* = r$. We have $(uv)^*Nr(uv) \subset p\mathbf{M}_n(M_1)p$. So, up to replacing v by uv , we may assume that $v^*z^\perp N z^\perp v \subset p\mathbf{M}_n(M_1)p$, $vv^* \in \mathcal{Z}(z^\perp N z^\perp)$ and $p = v^*v$.

This implies that $z^\perp N z^\perp v \subset v p\mathbf{M}_n(M_1)p$ and hence $Qz^\perp v \subset v p\mathbf{M}_n(M_1)p$. This further implies that $(Q \cap qM_1q)z^\perp v \subset v p\mathbf{M}_n(M_1)p$. Since $p = v^*v$, $vv^* \in \mathcal{Z}(z^\perp N z^\perp)$ and $Q \subset N$, the mapping $\rho : (Q \cap qM_1q)z^\perp \rightarrow p\mathbf{M}_n(M_1)p : x \mapsto v^*xv$ defines a unital normal $*$ -homomorphism such that $xv = v\rho(x)$ for all $x \in (Q \cap qM_1q)z^\perp$. Observe that $z \in M^\psi$ and hence $(Q \cap qM_1q)z^\perp \subset z^\perp M z^\perp$ is with expectation. By Proposition 2.7 (1), we obtain that $v \in (z^\perp M_1 \otimes \mathbf{M}_{1,n}(\mathbf{C}))p$ and hence $vv^* \in z^\perp (Q' \cap qMq) z^\perp = z^\perp (Q' \cap qM_1q) z^\perp$ and $Qz^\perp vv^* \subset vv^* z^\perp M_1 z^\perp vv^*$. Since $vv^* \leq z^\perp$, $vv^* \neq 0$ and $Q(z + vv^*) \subset (z + vv^*)M_1(z + vv^*)$, this contradicts the maximality of $z \in Q' \cap qM_1q$ and finishes the proof in the case when Q is of type III.

Since we have successively treated the cases when Q is of type II_1 , of type II_∞ and of type III, this finishes the proof of Lemma 4.4. \square

Lemma 4.5. *Let $q \in (M_1)^\psi$ be any nonzero projection and $Q \subset qMq$ any subfactor with separable predual that is not amenable and globally invariant under the modular automorphism group σ^{ψ_q} and such that $Q \cap qM_1q$ and $Q' \cap (qMq)^\omega$ are diffuse. Then $Q \subset qM_1q$.*

Proof. The proof, inspired by the one of [Ho12b, Theorem E], relies on Connes-Takesaki's structure theory [Co72, Ta03] and uses Corollary 4.2.

The novel aspect of the proof consists in combining [AH12, Theorem 4.1] and [MT13, Theorem 2.10] in order to obtain the following canonical inclusions of semifinite von Neumann algebras with trace preserving conditional expectations:

$$c_\phi(M) \subset c_{\phi^\omega}(M^\omega) \subset (c_\phi(M))^\omega$$

with $\phi = \varphi$ or $\phi = \psi$. More precisely, if we denote by $E_\omega^\phi : (c_\phi(M))^\omega \rightarrow c_\phi(M)$ the canonical faithful normal conditional expectation and by Tr_ϕ (resp. Tr_{ϕ^ω}) the canonical faithful normal semifinite trace on $c_\phi(M)$ (resp. $c_{\phi^\omega}(M^\omega)$), we have that $\text{Tr}_\phi \circ E_\omega^\phi$ is a faithful normal semifinite trace on $(c_\phi(M))^\omega$ and $(\text{Tr}_\phi \circ E_\omega^\phi)|_{c_{\phi^\omega}(M^\omega)} = \text{Tr}_{\phi^\omega}$. We will simply use the notation $\|\cdot\|_2$ for the L^2 -norm associated with any of the faithful normal semifinite traces considered above. We will use throughout the proof the identification $L_\phi(\mathbf{R}) = L_{\phi^\omega}(\mathbf{R}) \subset c_{\phi^\omega}(M^\omega)$.

Since $q \in M^\psi$ and $Q \subset qMq$ is globally invariant under the modular automorphism group σ^{ψ_q} , we may define $c_{\psi_q}(Q) = Q \rtimes_{\sigma^{\psi_q}} \mathbf{R}$ and regard $c_{\psi_q}(Q) \subset \pi_{\psi_q}(Q)c_\psi(M)\pi_{\psi_q}(Q)$ naturally. Fix an

arbitrary nonzero finite trace projection $r \in L_\psi(\mathbf{R})$ and put $\mathcal{M} = c_\varphi(M)$, $p = \Pi_{\varphi,\psi}(r) \in \mathcal{M}$, $\mathcal{Q} = \Pi_{\varphi,\psi}(rc_{\psi_q}(Q)r)$ and $\mathcal{P} = \mathcal{Q}' \cap (p\pi_\varphi(q)\mathcal{M}\pi_\varphi(q)p)^\omega$. Observe that

$$p\pi_\varphi(q) = \Pi_{\varphi,\psi}(r\pi_\psi(q)) = \Pi_{\varphi,\psi}(\pi_\psi(q)r) = \pi_\varphi(q)p$$

defines a nonzero projection in \mathcal{M} and is the unit of \mathcal{Q} .

Claim. We have $\mathcal{P} \not\preceq_{\mathcal{M}^\omega} (L_\varphi(\mathbf{R}))^\omega$.

Proof of the Claim. The proof uses an idea of [Io12, Lemma 9.5]. By contradiction, assume that $\mathcal{P} \preceq_{\mathcal{M}^\omega} (L_\varphi(\mathbf{R}))^\omega$. By [BHR12, Lemma 2.3], there exist $\delta > 0$ and a finite subset $\mathcal{F} \subset p\pi_\varphi(q)\mathcal{M}^\omega$ such that

$$(4.5) \quad \sum_{a,b \in \mathcal{F}} \|E_{(L_\varphi(\mathbf{R}))^\omega}(b^*ua)\|_2^2 > \delta, \forall u \in \mathcal{U}(\mathcal{P}).$$

For each $a \in \mathcal{F}$, write $a = (a_n)^\omega$ with a fixed sequence $(a_n)_n \in p\pi_\varphi(q)\mathcal{M}^\omega(\mathcal{M})$.

We next show that there exists $n \in \mathbf{N}$ such that

$$(4.6) \quad \sum_{a,b \in \mathcal{F}} \|E_{(L_\varphi(\mathbf{R}))^\omega}(b_n^*ua_n)\|_2^2 \geq \delta, \forall u \in \mathcal{U}(\mathcal{P}).$$

Assume by contradiction that this is not the case. Then for each $n \in \mathbf{N}$, there exists $u_n \in \mathcal{U}(\mathcal{P})$ such that

$$\sum_{a,b \in \mathcal{F}} \|E_{(L_\varphi(\mathbf{R}))^\omega}(b_n^*u_n a_n)\|_2^2 < \delta.$$

Since $p\pi_\varphi(q)\mathcal{M}\pi_\varphi(q)p$ is a finite von Neumann algebra, we may write $u_n = (u_m^{(n)})^\omega$ for some sequence $(u_m^{(n)})_m \in \ell^\infty(\mathbf{N}, p\pi_\varphi(q)\mathcal{M}\pi_\varphi(q)p)$ such that $u_m^{(n)} \in \mathcal{U}(p\pi_\varphi(q)\mathcal{M}\pi_\varphi(q)p)$ for all $m \in \mathbf{N}$. Then we have

$$\lim_{m \rightarrow \omega} \sum_{a,b \in \mathcal{F}} \|E_{L_\varphi(\mathbf{R})}(b_n^*u_m^{(n)}a_n)\|_2^2 < \delta.$$

Fix a $\|\cdot\|_2$ -dense countable subset $\{y_n : n \in \mathbf{N}\} \subset \mathcal{Q}$. Since $\lim_{m \rightarrow \omega} \|y_j u_m^{(n)} - u_m^{(n)} y_j\|_2 = \|y_j u_n - u_n y_j\|_2 = 0$ for all $n \in \mathbf{N}$ and all $0 \leq j \leq n$, we may choose $m_n \in \mathbf{N}$ large enough so that $v_n := u_{m_n}^{(n)} \in \mathcal{U}(p\pi_\varphi(q)\mathcal{M}\pi_\varphi(q)p)$ satisfies $\|y_j v_n - v_n y_j\|_2 \leq \frac{1}{n+1}$ for all $0 \leq j \leq n$ and $\sum_{a,b \in \mathcal{F}} \|E_{L_\varphi(\mathbf{R})}(b_n^*v_n a_n)\|_2^2 \leq \delta$. Since $p\pi_\varphi(q)\mathcal{M}\pi_\varphi(q)p$ is finite, we may define $v := (v_n)^\omega \in (p\pi_\varphi(q)\mathcal{M}\pi_\varphi(q)p)^\omega$. We moreover have $v \in \mathcal{U}(\mathcal{P})$ and

$$(4.7) \quad \sum_{a,b \in \mathcal{F}} \|E_{(L_\varphi(\mathbf{R}))^\omega}(b^*va)\|_2^2 = \lim_{n \rightarrow \omega} \sum_{a,b \in \mathcal{F}} \|E_{L_\varphi(\mathbf{R})}(b_n^*v_n a_n)\|_2^2 \leq \delta.$$

Equations (4.5) and (4.7) give a contradiction. This shows that Equation (4.6) holds. Therefore, up to replacing the finite subset $\mathcal{F} \subset p\pi_\varphi(q)\mathcal{M}^\omega$ by $\{a_n : a \in \mathcal{F}\} \subset p\pi_\varphi(q)\mathcal{M}$, we may assume that $\mathcal{F} \subset p\pi_\varphi(q)\mathcal{M}$ in Equation (4.5).

Since $Q' \cap (qMq)^\omega$ is diffuse and Q is globally invariant under the modular automorphism group σ^{ψ_q} , we know that $Q' \cap ((qMq)^\omega)^{\psi_q^\omega}$ is diffuse by [HR14, Theorem 2.3]. We may then choose a sequence $(u_n)_n \in \mathcal{M}^\omega(qMq)$ such that $(u_n)^\omega \in \mathcal{U}(Q' \cap ((qMq)^\omega)^{\psi_q^\omega})$ and $\lim_{n \rightarrow \infty} u_n = 0$ σ -weakly (see the first and second paragraphs in the proof of [HR14, Theorem A]). Observe that $(p\pi_\varphi(u_n)p)_n \in \ell^\infty(\mathbf{N}, p\pi_\varphi(q)\mathcal{M}\pi_\varphi(q)p)$ and

$$\begin{aligned} \pi_{\varphi^\omega}((u_n)^\omega)p &= \Pi_{\varphi^\omega, \psi^\omega}(\pi_{\psi^\omega}((u_n)^\omega)r) \\ &= \Pi_{\varphi^\omega, \psi^\omega}(r\pi_{\psi^\omega}((u_n)^\omega)r) \\ &= (\Pi_{\varphi, \psi}(r\pi_\psi(u_n)r))^\omega \\ &= (p\pi_\varphi(u_n)p)^\omega \in \mathcal{U}(\mathcal{P}). \end{aligned}$$

Since $\mathcal{F} \subset p\pi_\varphi(q)\mathcal{M}$, using Lemma 2.4 (with letting the Q there be the trivial algebra), we obtain $\lim_{n \rightarrow \omega} \|E_{L_\varphi(\mathbf{R})}(b^* p\pi_\varphi(u_n)pa)\|_2 = 0$ for all $a, b \in \mathcal{F}$. This implies that

$$(4.8) \quad \sum_{a,b \in \mathcal{F}} \|E_{(L_\varphi(\mathbf{R}))^\omega}(b^* \pi_\varphi^\omega((u_n)^\omega)pa)\|_2^2 = \lim_{n \rightarrow \omega} \sum_{a,b \in \mathcal{F}} \|E_{L_\varphi(\mathbf{R})}(b^* p\pi_\varphi(u_n)pa)\|_2^2 = 0.$$

Equation (4.5) with $\mathcal{F} \subset p\pi_\varphi(q)\mathcal{M}$ and Equation (4.8) give a contradiction. This finishes the proof of the Claim. \square

Next, for each $i \in \{1, 2\}$, put $\mathcal{M}_i = c_\varphi(M_i)$. We have $\mathcal{M} = \mathcal{M}_1 *_{L_\varphi(\mathbf{R})} \mathcal{M}_2$ (see [Ue98a, Theorem 5.1]). Observe that since M_1 is globally invariant under σ^ψ , we have $\Pi_{\varphi,\psi}(c_\psi(M_1)) = c_\varphi(M_1) = \mathcal{M}_1$. Since $r \in L_\psi(\mathbf{R}) \subset c_\psi(M_1)$, we have $p = \Pi_{\varphi,\psi}(r) \in \mathcal{M}_1$. Since $Q \cap qM_1q$ is diffuse and globally invariant under σ^{ψ_q} , we have $\Pi_{\varphi,\psi}(rc_{\psi_q}(Q \cap qM_1q)r) \not\leq_{\mathcal{M}} L_\varphi(\mathbf{R})$ by Lemma 2.5. Then [BHR12, Theorem 2.5] implies that $(\Pi_{\varphi,\psi}(rc_{\psi_q}(Q \cap qM_1q)r))' \cap p\pi_\varphi(q)\mathcal{M}\pi_\varphi(q)p \subset p\pi_\varphi(q)\mathcal{M}_1\pi_\varphi(q)p$ and hence

$$\mathcal{Q}' \cap p\pi_\varphi(q)\mathcal{M}\pi_\varphi(q)p = \mathcal{Q}' \cap p\pi_\varphi(q)\mathcal{M}_1\pi_\varphi(q)p.$$

The set of projections $s \in \mathcal{Q}' \cap p\pi_\varphi(q)\mathcal{M}\pi_\varphi(q)p = \mathcal{Q}' \cap p\pi_\varphi(q)\mathcal{M}_1\pi_\varphi(q)p$ such that $\mathcal{Q}s \subset s\mathcal{M}_1s$ attains its maximum in a unique projection z that belongs to $\mathcal{Z}(\mathcal{Q}' \cap p\pi_\varphi(q)\mathcal{M}\pi_\varphi(q)p) = \mathcal{Z}(\mathcal{Q}' \cap p\pi_\varphi(q)\mathcal{M}_1\pi_\varphi(q)p)$. Assume by contradiction that $z \neq p\pi_\varphi(q)$. Put $z^\perp := p\pi_\varphi(q) - z \in \mathcal{Z}(\mathcal{Q}' \cap p\pi_\varphi(q)\mathcal{M}\pi_\varphi(q)p)$. By assumption, we have $z^\perp \neq 0$.

Observe that since $Q \subset qMq$ is a subfactor that is not amenable, $\mathcal{Q} = \Pi_{\varphi,\psi}(rc_{\psi_q}(Q)r)$ has no amenable direct summand by [BHR12, Proposition 2.8]. By the previous Claim, we moreover have $\mathcal{P} \not\leq_{\mathcal{M}^\omega} (L_\varphi(\mathbf{R}))^\omega$. Then Corollary 4.2 implies that there exists $i \in \{1, 2\}$ such that $\mathcal{Q}z^\perp \leq_{\mathcal{M}} \mathcal{M}_i$. Hence, there exist $n \geq 1$, a finite trace projection $f \in \mathbf{M}_n(\mathcal{M}_i)$ (with respect to the canonical trace $\text{Tr}_{\varphi_i} \otimes \text{tr}_n$), a nonzero partial isometry $v \in (z^\perp \mathcal{M} \otimes \mathbf{M}_{1,n}(\mathbf{C}))f$ and a unital normal $*$ -homomorphism $\pi : \mathcal{Q}z^\perp \rightarrow f\mathbf{M}_n(\mathcal{M}_i)f$ such that $xv = v\pi(x)$ for all $x \in \mathcal{Q}z^\perp$. In particular, we have $\Pi_{\varphi,\psi}(rc_{\psi_q}(Q \cap qM_1q)r)v \subset v f\mathbf{M}_n(\mathcal{M}_i)f$. Since $\Pi_{\varphi,\psi}(rc_{\psi_q}(Q \cap qM_1q)r) \not\leq_{\mathcal{M}} L_\varphi(\mathbf{R})$, [BHR12, Theorem 2.5] and its Claim imply that $i = 1$ and $v \in (z^\perp \mathcal{M}_1 \otimes \mathbf{M}_{1,n}(\mathbf{C}))f$. Therefore we have $vv^* \in \mathcal{Q}' \cap p\pi_\varphi(q)\mathcal{M}_1\pi_\varphi(q)p$, $vv^* \neq 0$, $vv^* \leq z^\perp$ and $\mathcal{Q}(z + vv^*) \subset (z + vv^*)\mathcal{M}_1(z + vv^*)$. This contradicts the maximality of the projection $z \in \mathcal{Q}' \cap p\pi_\varphi(q)\mathcal{M}_1\pi_\varphi(q)p$.

Thus, we have $z = p\pi_\varphi(q)$ and hence

$$\Pi_{\varphi,\psi}(rc_{\psi_q}(Q)r) = \mathcal{Q} \subset p\pi_\varphi(q)\mathcal{M}_1\pi_\varphi(q)p = \Pi_{\varphi,\psi}(rc_{\psi_q}(qM_1q)r).$$

This implies that $rc_{\psi_q}(Q)r \subset rc_{\psi_q}(qM_1q)r$. Since this holds for every nonzero finite trace projection $r \in L_\psi(\mathbf{R})$, we obtain $c_{\psi_q}(Q) \subset c_{\psi_q}(qM_1q)$. Observe that $\pi_{\psi_q}(qMq) \subset c_{\psi_q}(qMq)$ is the fixed point algebra by an action of \mathbf{R} , called the dual action of σ^{ψ_q} , (see [Ta03, Theorem X.2.3 (i)]) and hence there exists a (non-normal) conditional expectation $F : c_{\psi_q}(qMq) \rightarrow \pi_{\psi_q}(qMq)$ such that $F(c_{\psi_q}(qM_1q)) = \pi_{\psi_q}(qM_1q)$. By applying the conditional expectation F to $\pi_{\psi_q}(Q) \subset c_{\psi_q}(Q) \subset c_{\psi_q}(qM_1q)$, we obtain $\pi_{\psi_q}(Q) \subset \pi_{\psi_q}(qM_1q)$ and hence $Q \subset qM_1q$. This finishes the proof of Lemma 4.5. \square

Proof of Theorem 4.3. Since both Q and $Q \cap M_1$ are with expectation in M , we may choose a faithful state $\psi \in M_*$ such that both Q and $Q \cap M_1$ are globally invariant under the modular automorphism group σ^ψ . Denote by $\mathbf{R} \rightarrow \mathcal{U}(M) : t \mapsto u_t = [D\psi : D\varphi]_t$ the Connes Radon-Nikodym cocycle (see [Co72, Théorème 1.2.1]) satisfying $\sigma_t^\psi = \text{Ad}(u_t) \circ \sigma_t^\varphi$ for all $t \in \mathbf{R}$.

Fix any $t \in \mathbf{R}$. Define the unital normal $*$ -isomorphism $\pi_t : Q \cap M_1 \rightarrow M : x \mapsto u_t^* x u_t$. Observe that

$$\pi_t(Q \cap M_1) = u_t^* Q \cap M_1 u_t = u_t^* \sigma_t^\psi(Q \cap M_1) u_t = \sigma_t^\varphi(Q \cap M_1) \subset \sigma_t^\varphi(M_1) = M_1$$

and $x u_t = u_t \pi_t(x)$ for all $x \in Q \cap M_1$. Since $Q \cap M_1 \subset M_1$ is diffuse and with expectation, Proposition 2.7 (1) implies that $u_t \in \mathcal{U}(M_1)$. Since this holds for every $t \in \mathbf{R}$, we obtain

$$\sigma_t^\psi(M_1) = u_t \sigma_t^\varphi(M_1) u_t^* = u_t M_1 u_t^* = M_1.$$

This implies that $\psi = \psi \circ E_{M_1}$ where $E_{M_1} : M \rightarrow M_1$ is the unique faithful normal conditional expectation.

Since $Q \cap M_1 \subset M_1$ is diffuse and with expectation, we have $Q' \cap M \subset (Q \cap M_1)' \cap M = (Q \cap M_1)' \cap M_1$ by Proposition 2.7 (1) and hence $Q' \cap M = Q' \cap M_1$. Denote by $z \in \mathcal{Z}(Q' \cap M^\omega)$ the unique central projection such that $(Q' \cap M^\omega)z$ is diffuse and $(Q' \cap M^\omega)z^\perp$ is atomic. By [HR14, Theorem 2.3], we have $z \in \mathcal{Z}(Q' \cap M) = \mathcal{Z}(Q' \cap M_1)$ and $(Q' \cap M^\omega)z^\perp = (Q' \cap M)z^\perp = (Q' \cap M_1)z^\perp$. Observe that $z \in (M_1)^\psi$.

Denote by $(z_n)_n$ a sequence of central projections in $\mathcal{Z}(Qz)$ such that $\sum_n z_n = z$, Qz_0 has a diffuse center and Qz_n is a diffuse factor for all $n \geq 1$. We have $\mathcal{Z}(Qz) \subset (Qz)' \cap zM^\psi z = z(Q' \cap M^\psi)z = z(Q' \cap (M_1)^\psi)z$. Moreover, since $\mathcal{Z}(Qz)z_0 \subset z_0 M_1 z_0$ is diffuse and globally invariant under the modular automorphism group $\sigma^{\psi_{z_0}}$, we have $Qz_0 \subset (\mathcal{Z}(Qz)z_0)' \cap z_0 M z_0 = (\mathcal{Z}(Qz)z_0)' \cap z_0 M_1 z_0$ by Proposition 2.7 (1). Finally, for all $n \geq 1$, since $Qz_n \subset z_n M z_n$ is a non type I subfactor that is globally invariant under the modular automorphism group $\sigma^{\psi_{z_n}}$ and such that $Qz_n \cap z_n M_1 z_n = (Q \cap M_1)z_n$ and $(Qz_n)' \cap (z_n M z_n)^\omega = (Q' \cap M^\omega)z_n$ are diffuse, Lemma 4.4, in the case when Qz_n is amenable, and Lemma 4.5, in the case when Qz_n is nonamenable, imply that $Qz_n \subset z_n M_1 z_n$. Therefore, we have $Qz \subset z M_1 z$. This finishes the proof of Theorem 4.3. \square

We can finally deduce the main results of this paper.

Proof of Theorem A. By applying Theorem 4.3 to the case when the projection $z \in \mathcal{Z}(Q' \cap M^\omega)$ satisfies $z = 1$, we obtain $Q \subset M_1$. \square

Proof of Corollary B. Since both Q and $Q \cap M_1$ are with expectation and $Q \cap M_1$ is diffuse, using Lemma 2.1, we may choose a faithful state $\psi \in M_*$ such that both Q and $Q \cap M_1$ are globally invariant under the modular automorphism group σ^ψ and the centralizer $(Q \cap M_1)^\psi$ is diffuse. Note that by the proof of Theorem 4.3, M_1 is also globally invariant under the modular automorphism group σ^ψ . Next, choose a diffuse abelian von Neumann subalgebra with separable predual $A \subset (Q \cap M_1)^\psi$.

Let $x \in Q$ be any element. Denote by $Q_0 \subset M$ the von Neumann subalgebra generated by the set $\{\sigma_t^\psi(y) : t \in \mathbf{R}, y = x \text{ or } y \in A\}$. Observe that $Q_0 \subset M$ has separable predual and is globally invariant under the modular automorphism group σ^ψ . Since Q is amenable and $Q_0 \subset Q$ is with expectation, it follows that Q_0 is also amenable. (It is true even in the non-separable case that amenability implies injectivity. See [Co76].) Since $A \subset (Q_0 \cap M_1)^\psi$ and since A is diffuse, $(Q_0 \cap M_1)^\psi$ is diffuse and so is $Q_0 \cap M_1$ (see e.g. [Bl06, Theorem IV.2.2.3]).

Since Q_0 is diffuse, amenable and with separable predual, the central sequence algebra $Q'_0 \cap Q_0^\omega$ is diffuse (see e.g. [Ho14, Proposition 2.6]). Since $Q_0 \subset M$ is with expectation, the inclusion $Q'_0 \cap Q_0^\omega \subset Q'_0 \cap M^\omega$ is with expectation and hence $Q'_0 \cap M^\omega$ is diffuse. Since $Q_0 \cap M_1$ is moreover diffuse and with expectation, we obtain that $Q_0 \subset M_1$ by Theorem A and hence $x \in M_1$. Since this holds true for all $x \in Q$, we deduce $Q \subset M_1$. \square

APPENDIX A. BICENTRALIZER PROBLEM FOR FREE PRODUCT VON NEUMANN ALGEBRAS

Let (M, φ) be any σ -finite von Neumann algebra endowed with a faithful normal state. Following [Ha85], the *asymptotic centralizer* of φ is defined by

$$\text{AC}(M, \varphi) = \left\{ (x_n)_n \in \ell^\infty(\mathbf{N}, M) : \lim_{n \rightarrow \infty} \|x_n \varphi - \varphi x_n\| = 0 \right\}$$

and the *bicentralizer* of φ is defined by

$$B(M, \varphi) = \left\{ a \in M : \lim_{n \rightarrow \infty} \|ax_n - x_na\|_\varphi = 0, \forall (x_n)_n \in AC(M, \varphi) \right\}.$$

Haagerup showed in [Ha85] that any amenable type III₁ factor with separable predual has trivial bicentralizer. It is an open problem, known as Connes's bicentralizer problem, to decide whether any type III₁ factor with separable predual has trivial bicentralizer.

It was recently showed in [HI15, Proposition 3.3] that $B(M, \varphi) = ((M^\omega)^{\varphi^\omega})' \cap M$ for every nonprincipal ultrafilter $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$. Using this characterization, we give a short proof of an unpublished result due to the second named author showing that Connes's bicentralizer problem has a positive solution for all type III₁ free product factors.

For each $i \in \{1, 2\}$, let (M_i, φ_i) be any nontrivial σ -finite von Neumann algebra endowed with a faithful normal state. Assume moreover that $\ker(\sigma^{\varphi_1}) \cap \ker(\sigma^{\varphi_2}) = \{0\}$. Denote by $(M, \varphi) = (M_1, \varphi_1) * (M_2, \varphi_2)$ the free product. By [Ue10, Theorem 4.1], we have $M = M_c \oplus M_d$ where M_c is a type III₁ factor and $M_d = 0$ or M_d is a multimatrix algebra. Put $\varphi_c = \frac{1}{\varphi(1_{M_c})}\varphi|_{M_c}$.

Theorem A.1. *Keep the same notation as above. Then $B(M_c, \varphi_c) = \mathbf{C}1_{M_c}$.*

Proof. In the case when both M_1 and M_2 are atomic, φ_c is an almost periodic state such that $((M_c)^{\varphi_c})' \cap M_c^\omega = \mathbf{C}1_{M_c}$ by [Ue11, Theorem 2.2]. Then we have $B(M_c, \varphi_c) \subset ((M_c)^{\varphi_c})' \cap M_c = \mathbf{C}1_{M_c}$.

Next, we may assume that M_1 has a diffuse direct summand. Since M_c is of type III and using [Ue10, Lemma 2.2], up to cutting down M by the central projection in M_1 that supports the diffuse direct summand of M_1 , we may assume without loss of generality that M_1 is diffuse. In that case, we have $M = M_c$. Observe that M_1^ω and M_2^ω are both globally invariant under the modular automorphism group σ^{φ^ω} and are $*$ -free inside M^ω with respect to the state φ^ω (see [Ue00, Proposition 4]). Letting $P = M_1^\omega \vee M_2^\omega$, we have $(P, \varphi^\omega|_P) \cong (M_1^\omega, \varphi_1^\omega) * (M_2^\omega, \varphi_2^\omega)$ and $M \subset P \subset M^\omega$.

Since M_1 is diffuse and $M_2 \neq \mathbf{C}1$, we have that $(M_1^\omega)^{\varphi_1^\omega}$ is diffuse and $(M_2^\omega)^{\varphi_2^\omega} \neq \mathbf{C}1$ by Proposition 2.8. Using [HI15, Proposition 3.3] and Proposition 2.7 (1), we have

$$B(M, \varphi) = ((M^\omega)^{\varphi^\omega})' \cap M \subset ((M_1^\omega)^{\varphi_1^\omega})' \cap P \cap M \subset ((M_1^\omega)^{\varphi_1^\omega})' \cap M_1^\omega \cap M \subset M_1.$$

Next, one can choose an invertible element $w \in (M_2^\omega)^{\varphi_2^\omega}$ such that $\varphi_2^\omega(w) = 0$. For all $y \in B(M, \varphi) \subset M_1$ such that $\varphi_1(y) = 0$, using the freeness with respect to φ^ω and since $yw = wy$, we have

$$\varphi^\omega(w^*y^*yw) = \varphi^\omega(w^*y^*wy) = 0.$$

Therefore $w^*y^*yw = 0$ and hence $y = 0$ since w is invertible. It immediately follows that $B(M, \varphi) = \mathbf{C}1$. This finishes the proof of Theorem A.1. \square

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